

Nonholonomic Jet Deformations, Exact Solutions for Modified Ricci Soliton and Einstein Equations

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Abstract

Let \mathbf{g} be a pseudo-Riemannian metric of arbitrary signature on a manifold \mathbf{V} with conventional $n + n$ dimensional splitting, $n \geq 2$, determined by a nonholonomic (non-integrable) distribution \mathcal{N} defining a generalized (nonlinear) connection and associated nonholonomic frame structures. We work with an adapted linear metric compatible connection $\hat{\mathbf{D}}$ and its nonzero torsion $\hat{\mathbf{T}}$, both completely determined by \mathbf{g} . Our first goal is to prove that there are certain generalized frame and/or jet transforms and prolongations with $(\mathbf{g}, \mathbf{V}) \rightarrow (\hat{\mathbf{g}}, \hat{\mathbf{V}})$ into explicit classes of solutions of some generalized Einstein equations $\hat{\mathbf{R}}ic = \Lambda \hat{\mathbf{g}}$, $\Lambda = const$, encoding various types of (nonholonomic) Ricci soliton configurations and/or jet variables and symmetries. The second goal is to solve additional constraint equations for zero torsion, $\hat{\mathbf{T}} = 0$, on generalized solutions constructed in explicit forms with jet variables and extract Levi-Civita configurations. This allows us to find generic off-diagonal exact solutions depending on all space time coordinates on \mathbf{V} via generating and integration functions and various classes of constant jet parameters and associated symmetries. Our third goal is to study how such generalized metrics and connections can be related by the so-called "half-conformal" and/or jet deformations of certain sub-classes of solutions with one, or two, Killing symmetries. Finally, we present some examples of exact solutions constructed as nonholonomic jet prolongations of the Kerr metrics, with possible Ricci soliton deformations, and characterized by nonholonomic jet structures and generalized connections.

Keywords: Nonholonomic manifolds and jets, generalized connections, geometric methods and PDE, Ricci solitons, Einstein manifolds, modified gravity, exact solutions and mathematical relativity.

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1 Introduction

Various results and methods of the theory of nonholonomic manifolds, jets and connections can be combined and applied to the study of symmetries of systems of nonlinear partial differential equations, PDEs, and constructing exact and approximate solutions. In modern physics, such fundamental field and evolution equations are related to the Ricci soliton geometry, mathematical relativity, particle physics and geometric mechanics [36, 38, 39, 40]. For instance, a jet space technique was elaborated upon to analyze special features of the vacuum Einstein equations in general relativity, GR, that allows to define certain generalized symmetries and

conservation laws. In a more general context, a Lagrangian formalism was elaborated on the jet–gauge and jet–diffeomorphism groups with the aim of unifying gravity with internal gauge symmetries [1]. Another direction related to Finsler–Lagrange geometry and nonholonomic mechanics was considered by the authors of papers [4, 30, 2, 3], where certain generalizations of Einstein equations were formulated on jet spaces endowed with nonlinear connection structures.

In recent years, series of works have been devoted to elucidating geometric methods that allow for the decoupling of (modified) Einstein equations for certain "auxiliary" connections with respect to adapted nonholonomic frames, and constructing generic off–diagonal solutions¹ depending on all spacetime coordinates, see reviews of results in [40, 45, 15]. Following the so–called anholonomic frame deformation method, AFDM, the solutions are generated in explicit forms via formulae determined by generating and integration functions, and various commutative and noncommutative parameters. Such solutions may exhibit Killing, non–Killing solitonic and/or other type of symmetries, which for respective boundary/ initial / source conditions can be with nontrivial space–time topology. The solutions may also describe evolution and/or dynamical processes, or result in stochastic behaviour. We can extract Levi–Civita configurations with zero torsion if we impose additional nonholonomic constraints on certain classes of generalized solutions. It should be noted that because such systems are nonlinear it is important to consider the restrictions via integration / generation functions and constants, symmetry / boundary / initial conditions "at the end", on some defined integral varieties. By prescribing from the very beginning only some special ansatz for the metrics and connections may result in a simplified system of equations (for instance, to transform it into a nonlinear system of ordinary differential equations), that may not decouple the PDEs in a general form. This can result in reducing the number of the bulk of nonlinear off–diagonal multi-variables.

The goal of this work is to study the basic properties of nonholonomic Ricci soliton and (modified) Einstein equations with metrics and (generalized) connections generated by jet prolongations of exact solutions. We study also the constraints under which various classes of solutions with generalized jet variables and symmetries are transformed into standard Einstein metrics with jet parametric dependence of generic off–diagonal metrics. Readers are referred to monographs [36, 23] on main results on jets and jet bundle geometry. The literature on nonholonomic jet manifolds and bundles is less popular and more sophisticated than that on holonomic jets. Experts on mathematical relativity and PDEs are less familiar with the geometry of nonholonomic manifolds elaborated as in the Vranceanu–Horak approach [47, 48, 49, 22], see recent results and applications in [6, 44]. We cite here some important works on generalized connections developed by different schools of differential geometry on nonholonomic jets, quasi-jets and the theory of higher order connections, see [12, 34, 37, 10, 46]. We will sketch a few essential notions and necessary results using recent approaches formulated in Refs. [26, 27].

In this work, we follow three explicit goals motivated and stated in section 2.4.5. The first goal is to develop the anholonomic frame deformation methods, (AFDM, see reviews of results in [40, 43, 45]) in such a form that will allow us to decouple the nonholonomic r –jet deformations of the Ricci soliton and Einstein equations, and integrate such equations for general classes of generic off–diagonal metric and nonlinear connection structures. The second goal is to show how we can extract from extra dimensional jet configurations the Levi–Civita connections (in particular, physically important solutions in GR with jet parameters) by solving the nonholonomic constraints for zero torsion conditions. Finally, the third goal is to analyze explicit examples of exact solutions depending on jet parameters defining nonholonomic deformations of black hole solutions and gravitational solitonic waves. We study how nonholonomic and/or r –jet deformations of the Kerr metric may model the physical effects of Ricci solitons in massive gravity and other modified gravity models [8, 31, 32, 14, 35, 19, 20, 24, 5, 28, 15].

The article is organized as follow. In section 2, we recall basic facts and definitions concerning nonholonomic manifolds and jets and elaborate on the concept of generalized connection structure. We provide an introduction to the geometry of nonholonomic manifolds and bundles endowed with nonlinear connection structures. There are outlined main results and stated respective denotations on nonholonomic maps and jets of (non) holonomic manifolds. *The first important result is formulated in Theorem 2.3* stating that there is a canonical distinguished connection structure on r –jet prolongation of (modified) Einstein manifolds which will allow to prove the main

¹which can not be diagonalized by coordinate transforms in a finite spacetime region, for instance, in GR

results in the next section. Then, we elaborate in details the formalism of nonholonomic r -jet prolongation of Ricci soliton and (generalized/modified) equations. This allows us to formulate and prove *the second important result, i.e. Theorem 2.4*, which provides the N-adapted equations for gradient canonical Ricci jet-solitons and generalizes the jet-extensions of Einstein manifolds.

In section 3, we formulate and prove *the main theorems (the first two main results of this work) on decoupling, see Theorem 3.1, and integration, see Theorem 3.2*, of (modified) Ricci soliton and Einstein equations. The approach consists of the generalization of the results for nonholonomic jet prolongations of fundamental geometric and physical objects in generalized/ modified gravity theories and further developments of AFDM. The key idea is to consider nonholonomic $2+2+2+\dots$ splitting with two dimensional (2-d) shells of jet coordinates and adapting the geometric constructions for such nonholonomic spacetime and jet distributions.

Section 4 is devoted to explicit examples of exact solutions depending on jet coordinates, jet parameters, symmetries, Killing and non-Killing symmetries, deformations by Ricci soliton configurations, modified gravity contributions, mimicking massive gravity terms with effective cosmological constant and gravitational polarizations. *The third main result of this paper, Theorem 4.1*, is related to Ricci soliton modifications and r -jet prolongations of the Kerr metric which play an essential role in the physics of black holes. Such black hole metrics can be extended to generic off-diagonal forms for various classes of modified gravity theories with extra dimensions [40, 43, 45, 15]. The AFDM even allows us to construct very general integral varieties for such gravitational and geometric evolution like nonlinear systems of PDEs. For r -jet configurations, it is clear that new classes of gravitational and matter field equations at least possess certain jet type local symmetries and possible association with nonlinear gauge interior degrees of freedom. We show that such solutions can be constructed both with zero or non-zero canonical torsion, with possible rotoid symmetries for Kerr – de Sitter configurations and other classes of vacuum and non-vacuum jet prolongations.

In Appendix, we provide a summary of the most important and necessary N-adapted coefficient formulas and provide technical details of some theorems.

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2 Nonholonomic Manifolds, Jet Bundles and Generalized Connections

We start by recalling a few basic definitions on the geometry of nonholonomic manifolds and bundles, related jet spaces and theory of generalized (nonlinear) connections [36, 23, 26, 27]. The geometric approach is generalized in a form to unify both the concepts of nonholonomic manifolds [47, 48, 49, 22, 6, 44] and that of nonholonomic jet spaces [12, 34, 27].

We shall work in the category of $n + m$ dimensional nonholonomic manifolds \mathcal{V} , with $n, m \geq 2$, of necessary smooth class (for instance, of class \mathcal{C}^∞), Hausdorff, finite dimensional and without boundaries. The solutions of certain systems of nonlinear partial differential equations (PDE) can be topologically nontrivial, with singularities and various type of Killing and non-Killing symmetries. Such PDEs, nonholonomic constraints² and their solutions are for geometric models of (modified) gravity theories and Ricci soliton equations defined as certain stationary configurations in a nonholonomic geometric evolution system, with possible Wick rotations (for small deformations) and frame transformations between Lorentzian and Euclidean signatures of metrics.

2.1 Holonomic jets

Jets are certain equivalence classes of smooth maps between two manifolds M , $\dim M = n$, and Q , $\dim Q = m$, when the maps are represented by Taylor polynomials. One writes this as $f, g : M \rightarrow Q$ and says that a r -jet is determined at a point $u \in M$ if there is a r -th order contact at u . The idea is formalized mathematically using the concept as the r -th order contact of two curves on a manifold.

²equivalently, anholonomic (nonholonomic), i.e. non-integrable

Definition 2.1 -Lemma: Two curves $\gamma, \delta : \mathbb{R} \rightarrow V$ have the r -th contact at zero if for every smooth function φ on M the difference $\varphi \circ \gamma - \varphi \circ \delta$ vanishes to r -th order at $0 \in \mathbb{R}$. In this case, we have an equivalence relation $\gamma \sim_r \delta$ when $r = 0$ implies $\gamma(0) = \delta(0)$. If $\gamma \sim_r \delta$, then $f \circ \gamma \sim_r f \circ \delta$ for every map $f : b \rightarrow Q$

Two maps $f, g : V \rightarrow Q$ are said to determine the same r -jet at $x \in M$, if for every curve $\gamma : \mathbb{R} \rightarrow V$ with $\gamma(0) = x$ the curves $f \circ \gamma$ and $g \circ \gamma$ have the r -th order contact at zero. In such a case, we write $j_x^r f = j_x^r g$, or $j^r f(x) = j^r g(x)$. An equivalence class of this relation is called an r -jets of M into Q .

Definition 2.2 The set of all r -jets of M into Q is denoted by $J^r(M, Q)$; for an element $X = j_x^r f \in J^r(M, Q)$, the point $x := \alpha X$ is the source of X and the point $f(x) =: \beta X$ is the target of X .

One denotes by $\pi_s^r, 0 \leq s \leq r$ the projection $j_x^r f \rightarrow j_x^s f$ of r -jets into s -jets. All r -jets form a category, the units of which are the r -jets of the identity maps of manifolds.

By $J_x^r(M, Q)$, or $J_x^r(M, Q)_y$ we mean the set of all r -jets of x onto Q with source $x \in M$, or tangent $y \in Q$,

$$J_x^r(M, Q)_y = J_x^r(M, Q) \cap J_x^r(M, Q)_y \text{ and } L_{n,m}^r = J_0^r(\mathbb{R}^n, \mathbb{R}^m)_0$$

In local coordinates x^i , the value $\partial_{\check{i}} f := \frac{\partial^{\check{i}} f}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}$ is the partial derivative of a function $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, with a **multi-index** \check{i} of range n , which is a m -tuple $\check{i} = (i_1, \dots, i_n)$ of non-negative integers. We write $|\check{i}| = i_1 + \dots + i_n$, with $\check{i}! = i_1! i_2! \dots i_n!$, $0! = 1$, and $x^{\check{i}} = (x^1)^{i_1} \dots (x^n)^{i_n}$ for $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$.³

Consider a local coordinate system x^i on M and a local coordinate system y^a on Q . Two maps $f, g : M \rightarrow Q$ satisfy $j_x^r f = j_x^r g$ if and only if all the partial derivatives up to order r of the components f^a and g^a of their coordinate expressions coincide at x . In this a case the chain rule implies $f \circ \gamma \sim_r g \circ \gamma$. For the curves $x^i = \zeta^i t$ with arbitrary ζ^i , these conditions read $\sum_{|\check{i}|=k} (\partial_{\check{i}} f^a(x)) \zeta^{\check{i}} = \sum_{|\check{i}|=k} (\partial_{\check{i}} g^a(x)) \zeta^{\check{i}}$, for $k = 0, 1, \dots, r$.

The elements of $L_{n,m}^r$ can be identified with the r -th order Taylor expansions of the generating maps, i.e. with m -tuples of polynomials of degree r in m variables without the absolute term. Such an expression $\sum_{1 \leq |\check{i}| \leq r} \zeta_i^a x^{\check{i}}$ is the polynomial representative of a r -jet. Hence $L_{n,m}^r$ is a numerical space of the variables ζ_i^a .

Standard combinatorics yields $\dim L_{n,m}^r = m \left[\binom{n+r}{n} - 1 \right]$. The coordinates on $L_{n,m}^r$ are sometimes denoted more explicitly by $\zeta_i^a, \zeta_{ij}^a, \dots, \zeta_{i_1 \dots i_r}^a$, symmetric in all subscripts. The projection $\pi_s^r : L_{m,n}^r \rightarrow L_{m,n}^s$ consists in suppressing all terms of degree $> s$.

The set of all invertible elements of $L_{n,m}^r$ with the jet composition is a Lie group G_m^r called the r -th differential group of the r -th jet group in dimension m . For $r = 1$, the group G_m^1 is identified with $GL(m, \mathbb{R})$.

Let $p : Q \rightarrow M$ be a fibered manifold.

Definition 2.3 A map $j^r f : M \rightarrow J^r(M, Q)$ is called a r -th jet prolongation of $f : M \rightarrow Q$. The set $J^r Q$ of all r -jets of the local sections of Y is called the r -th jet prolongation of Q and $J^r Q \subset J^r(M, Q)$ is a closed submanifold.

We note that if $Q \rightarrow M$ is a vector bundle, then $J^r Q$ is a also a vector bundle.

2.2 Nonholonomic manifolds and nonlinear connections

The concept of nonholonomic jet is elaborated in Refs. [12, 34, 27], when multi-indices are not symmetric and the jet spaces are subject to certain non-integrable conditions. Nonholonomic structures with non-integrable constraints can be defined not only on the space of jets but also on the 'prime', M , and 'target', Q , manifolds. In our approach, we shall elaborate a geometric formalism encoding nonholonomic geometric structures both on manifolds and maps, i.e. on M, Q and $J^r(M, Q)$.

³Our definition of multi-index derivative $\partial_{\check{i}} f$ is similar to $D_i f$ used in [23]. For clarity, we need to modify the system of notations in order to elaborate a geometric method of constructing exact solutions of PDEs with jet variables.

By definition, a *nonholonomic manifold* \mathbf{V} is a manifold endowed with a *nonholonomic distribution*. In this work we follow the approach elaborated by G. Vrăncanu [47, 48, 49] and Z. Horak [22], see reviews [6, 44]. For our purposes (to construct jet-generalizations of the Einstein equations and physically relevant solutions), it is enough to consider a nonholonomic distribution determined by a *nonlinear connection* (N-connection) structure $\mathbf{N} = \{N_i^a(x, y)\}$. Such a N-connection can be introduced as a Whitney sum

$$\mathbf{N} : T\mathbf{V} := h\mathbf{V} \oplus v\mathbf{V}, \quad (1)$$

where $T\mathbf{V}$ is the tangent bundle of \mathbf{V} and $h\mathbf{V}$ and $v\mathbf{V}$ are, respectively the horizontal (h) and vertical (v) subspaces for a nonholonomic fibration.⁴ N-connections were used in coordinate form by E. Cartan in his model of Finsler geometry [9] by considering $\mathbf{V} = TM$ as a tangent bundle to a manifold M . In a similar form, we can work with a vector bundle, $V = E$, on M , $\dim E = n + m$, $\dim M = n$ (for $n, m \geq 2$) instead of TM . The global definition of N-connection is due to C. Ehresmann [11]. In [23], such connections are studied for fiber bundles and are called generalized (Ehresmann) connections.⁵ We will follow a different system of notations that were elaborated upon and used in the theory of nonholonomic (non) commutative Ricci flows, nonholonomic Dirac operators and Clifford bundles, and, deformations and quantization of generalized geometries and gravity theories [41, 42, 44].

Any \mathbf{N} defines a N-adapted frame structure $\mathbf{e}_\alpha = (\mathbf{e}_i, e_a)$, on $T\mathbf{V}$, and co-frame structure $\mathbf{e}^\beta = (e^j, \mathbf{e}^b)$, on the dual tangent bundle $T^*\mathbf{V}$,

$$\mathbf{e}_\alpha = (\mathbf{e}_i = \partial_i - N_i^b \partial_a, e_a = \partial_a) \text{ and } \mathbf{e}^\beta = (e^j = dx^j, \mathbf{e}^b = dy^b + N_i^b dx^i), \quad (2)$$

where the Einstein summation convention is applied on repeated indices and $\partial_i = \partial/\partial x^i$ and $\partial_a = \partial/\partial y^a$. In general, such local bases are nonholonomic, i.e. $\mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma$, with nontrivial nonholonomy coefficients $W_{\alpha\beta}^\gamma$. In this work, we take these frame structures to be canonical in the sense that they are linear on N-connection coefficients and admit (see Theorem 3.1) the decoupling of (modified) Einstein equations in general form. Here we note that although there are different canonical N-connection structures in different models of Finsler-Lagrange geometry, Hamilton geometry etc., see details in [9, 41, 42, 44], those constructions can not be used for constructing exact solutions in gravity theories. We call certain geometric objects to be distinguished objects (d-objects), for instance d-tensors, d-vectors if they are determined by the coefficients in N-adapted form⁶, i.e. with respect to N-elongated (co) bases (2) and their tensor products. For instance, a vector $X \in T\mathbf{V}$ can be written in a "non N-adapted" coordinate form, $X = X^\alpha \partial_\alpha$, or as a d-vector, $\mathbf{X} = hX \oplus vX = \mathbf{X}^\alpha \mathbf{e}_\alpha = X^i \mathbf{e}_i + X^a e_a$.

Two important characteristics of a N-connection are 1) the almost complex structure \mathbf{J} , where $\mathbf{J}(\mathbf{e}_i) = -e_{2+i}$ and $\mathbf{J}(e_{2+i}) = \mathbf{e}_i$, with \mathbf{J} satisfying the symplectic relation $\mathbf{J} \circ \mathbf{J} = -\mathbb{I}$, where \mathbb{I} is the unit matrix and 2) the Neijenhuis tensor (called also the curvature of N-connection) defined as

$${}^N\mathbf{J}[\mathbf{X}, \mathbf{Y}] := -[\mathbf{X}, \mathbf{Y}] + [\mathbf{J}\mathbf{X}, \mathbf{J}\mathbf{Y}] - \mathbf{J}[\mathbf{J}\mathbf{X}, \mathbf{Y}] - \mathbf{J}[\mathbf{X}, \mathbf{J}\mathbf{Y}], \quad \forall \mathbf{X}, \mathbf{Y} \in T\mathbf{V}.$$

Linear connections on (\mathbf{V}, \mathbf{N}) can be defined in N-adapted form as distinguished connections, *d-connections*, in order to preserve under parallel transport the distribution (1). Such a covariant differential operator splits as $\mathbf{D} = (hD, vD)$. We can associate to \mathbf{D} a 1-form $\mathbf{\Gamma}_\alpha^\gamma = \mathbf{\Gamma}_{\alpha\beta}^\gamma \mathbf{e}^\beta$ and elaborate a N-adapted differential form calculus. The torsion and curvature are defined, respectively, in terms of standard formulae:

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) := \mathbf{D}_\mathbf{X} \mathbf{Y} - \mathbf{D}_\mathbf{Y} \mathbf{X} + [\mathbf{X}, \mathbf{Y}] \quad \text{and} \quad \mathbf{R}(\mathbf{X}, \mathbf{Y}) := \mathbf{D}_\mathbf{X} \mathbf{D}_\mathbf{Y} - \mathbf{D}_\mathbf{Y} \mathbf{D}_\mathbf{X} - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]}. \quad (3)$$

⁴Local coordinates are with a conventional 2+2 splitting, $u^\alpha = (x^i, y^a)$, with $i, j, \dots = 1, 2$ and $a, b, \dots = 3, 4, \dots$; in brief, $u = (x, y) \in \mathbf{V}$ for any point and its coordinates. We shall use boldface symbols in order to emphasize that certain spaces and/or geometric objects are provided with or adapted to a N-connection structure.

⁵We refer the readers to this monograph for a modern approach to differential geometry and main results on jets, Weil bundles and generalized connections.

⁶this mean that certain geometric constructions are adapted to a horizontal h - and vertical v -splitting stated by a N-connection distribution (1)

Also, in the usual way, the Ricci d-tensor \mathbf{Ric} is constructed by the contraction of indices in the curvature tensor $\mathbf{R} = \{\mathbf{R}^\alpha_{\beta\gamma\mu}\}$, $\mathbf{Ric} := \{\mathbf{R}_{\beta\gamma} = \mathbf{R}^\alpha_{\beta\gamma\alpha}\}$, [41, 42, 44]. Readers may study such papers, and references therein, on deformation quantization of gravity based on almost complex structures characterizing generic off-diagonal solutions.

Let \mathbf{g} be a metric of arbitrary signature on a nonholonomic manifold/ bundle (\mathbf{V}, \mathbf{N}) which in \mathbf{N} -adapted form (1) is represented as a symmetric d-tensor,

$$\mathbf{g} = hg \oplus vh = \mathbf{g}_{\alpha\beta}(u)\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}(x, y)dx^i \otimes dx^j + g_{ab}(x, y)\mathbf{e}^a \otimes \mathbf{e}^b.$$

For any metric structure \mathbf{g} on a nonholonomic manifold (\mathbf{V}, \mathbf{N}) , there are two "preferred" linear connections, completely and uniquely, defined by

$$\mathbf{g} \rightarrow \begin{cases} \nabla : & \nabla \mathbf{g} = \mathbf{0}; \nabla \mathbf{T} = 0, & \text{the Levi-Civita connection;} \\ \hat{\mathbf{D}} : & \hat{\mathbf{D}} \mathbf{g} = \mathbf{0}; h\hat{\mathbf{T}} = 0, v\hat{\mathbf{T}} = 0, & \text{the canonical d-connection.} \end{cases} \quad (4)$$

It should be noted that ∇ is not a d-connection because its parallel transport does not preserve the horizontal h - and the vertical v -splitting (1). Nevertheless, there is a unique \mathbf{N} -adapted distortion relation

$$\hat{\mathbf{D}} = \nabla + \hat{\mathbf{Z}} \quad (5)$$

when both linear connections $\hat{\mathbf{D}}$ and ∇ and the distorting d-tensor $\hat{\mathbf{Z}}$ are completely determined by the metric structure \mathbf{g} for a prescribed \mathbf{N} -connection structure \mathbf{N} . The Ricci and Riemannian tensors are different for $\hat{\mathbf{D}}$ and ∇ because, in general, $\hat{\mathbf{T}} \neq 0$ but $\nabla \mathbf{T} = 0$. All geometric constructions with $(\mathbf{g}, \nabla; \mathbf{V})$ can be transformed equivalently into similar ones with $(\mathbf{g}, \mathbf{N}, \mathbf{D}; \mathbf{V})$, and conversely, if distortion relations (5) are utilized.

There are two canonical scalars determined by a d-metric \mathbf{g} via $\hat{\mathbf{D}}$, ${}^s\hat{\mathbf{R}} := \mathbf{g}^{\beta\gamma}\hat{\mathbf{R}}_{\beta\gamma}$ and the standard (pseudo) Riemannian scalar determined by ∇ , $R := \mathbf{g}^{\beta\gamma}R_{\beta\gamma}$. Both values are related by a distortion relation which can be found by contracting with $\mathbf{g}^{\beta\gamma}$ nonholonomic deformations of the Ricci tensor, $\hat{\mathbf{Ric}} = \mathbf{Ric} + \hat{\mathbf{Z}}ic$, which are computed by substituting (5) in formulae (3).

2.3 Nonholonomic jets and \mathbf{N} -adapted manifolds and maps

Nonholonomic jet structures can be introduced even if the prime and target manifolds are considered only with holonomic distributions. In a more general context, all maps and manifolds can be nonholonomic.

2.3.1 Nonholonomic maps of holonomic manifolds

Let us consider two holonomic manifolds M and Q and introduce the set of nonholonomic 1-jets $\mathbf{J}^1(M, Q) := J^1(M, Q)$ for $r = 1$.⁷ By induction, we can consider the source projection $\alpha : \mathbf{J}^{r-1}(M, Q) \rightarrow M$ and the target projection $\beta : \mathbf{J}^{r-1}(M, Q) \rightarrow Q$ as the target projection of $(r - 1)$ -th nonholonomic jets.

Definition 2.4 *An $\mathcal{X} \in \mathbf{J}^r(M, Q)$ is said to be a nonholonomic r -jet with the source $x \in M$ and the target $y \in Q$ if there is a local section $\sigma : M \rightarrow \mathbf{J}^r(M, Q)$ such that $\mathcal{X} = \mathbf{j}_x^1\sigma$ and $\beta(\sigma(x)) = y$*

We write $\mathcal{X} = \mathbf{j}_x^1\sigma$ (with calligraphic \mathcal{X}) instead of $X = \mathbf{j}_x^1\sigma$ from Definition 2.2 in order to emphasize that the jet map is defined, in general, in nonholonomic form. There is a natural embedding $J^r(M, Q) \subset \mathbf{J}^r(M, Q)$. In general, any \mathcal{X} induces a nonholonomic map $\mu\mathcal{X} : (\underbrace{TT \dots T}_{r\text{-times}} M)_x \rightarrow (\underbrace{TT \dots T}_{r\text{-times}} Q)_y$, [27].

⁷In [27], this is written $\tilde{J}^1(M, Q)$ instead of boldface $\mathbf{J}^1(M, Q)$. As we mentioned above, we use boldface letters in order to emphasize horizontal h - and vertical v -splittings via a \mathbf{N} -connection structure of a class of geometric objects/ maps / spaces. We can consider such decompositions from the maps defining a jet structure (and write \mathbf{J}^1) even when the respective prime and target manifold are holonomic ones, when M and Q are not boldface.

2.3.2 Nonholonomic maps of nonholonomic manifolds

We can generalize the constructions with nonholonomic jets by considering that the geometric objects and transforms are defined by equivalence classes of smooth maps between two nonholonomic manifolds $\mathbf{V}, \dim \mathbf{V} = n + n$, and $\mathbf{Q}, \dim \mathbf{Q} = m + m$, and such maps are represented by Taylor polynomials in certain N-adapted local frames. Other types of nonholonomic geometric models can also be elaborated on in a similar manner for N-adapted maps of type $\mathbf{V} \rightarrow \mathbf{V}'$, where $\dim \mathbf{V} = n + m$ and $\dim \mathbf{V}' = n' + m'$ and there are defined N-connection decompositions $T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}$ and $T\mathbf{V}' = h'\mathbf{V}' \oplus v'\mathbf{V}'$ of type (1) with corresponding mappings $\mathbf{N} \rightarrow \mathbf{N}'$.

Definition 2.5 An $\mathcal{X} \in \mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ is said to be a complete nonholonomic r -jet with the source $\mathbf{u} \in \mathbf{V}$ and the target $\mathbf{u}' \in \mathbf{V}'$ if there is a local section $\sigma : \mathbf{V} \rightarrow \mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ such that $\mathcal{X} = \mathbf{j}_{\mathbf{u}}^1 \sigma$ and $\beta(\sigma(\mathbf{u})) = \mathbf{u}'$.

For simplicity, we use the same nonholonomic jet symbol $\mathcal{X} = \mathbf{j}_{\mathbf{u}}^1 \sigma$ with boldface point $\mathbf{u} \in \mathbf{V}$. There are also defined natural embeddings $J^r(V, V') \subset \mathbf{J}^r(V, V') \subset \mathbf{J}^r(\mathbf{V}, \mathbf{V}')$, that can be parameterized by local coordinate and/or N-adapted frame systems and integrable or non-integrable maps. In general, any \mathcal{X} induces a nonholonomic map $\mu\mathcal{X} : (\underbrace{TT \dots T}_{r\text{-times}} \mathbf{V})_{\mathbf{u}} \rightarrow (\underbrace{T'T' \dots T'}_{r\text{-times}} \mathbf{V}')_{\mathbf{u}'}$ that splits into horizontal and vertical components

with $h, v, \dots \rightarrow h', v', \dots$

We can generalize the concept of jet prolongation of fibered manifold, see Definition 2.3, to cases with nonholonomic maps and to prime and target nonholonomic manifolds. Let $\mathbf{p} : \mathbf{Q} \rightarrow \mathbf{V}$ be a fibered manifold when, in general, both \mathbf{Q} and \mathbf{V} are with nontrivial N-connection structures.

Definition 2.6 A nonholonomic map $\mathbf{j}^r \mathbf{f} : \mathbf{V} \rightarrow \mathbf{J}^r(\mathbf{V}, \mathbf{Q})$ is called a r -th jet prolongation of $\mathbf{f} : \mathbf{V} \rightarrow \mathbf{Q}$. The set $\mathbf{J}^r \mathbf{Q}$ of all r -jets of the local sections of \mathbf{Q} is called the r -th jet prolongation of \mathbf{Q} and $\mathbf{J}^r \mathbf{Q} \subset \mathbf{J}^r(\mathbf{V}, \mathbf{Q})$ is a closed submanifold.

We note that if $\mathbf{Q} \rightarrow \mathbf{V}$ is a distinguished vector bundle with nonholonomic base and nonholonomic total spaces, then $\mathbf{J}^r \mathbf{Q}$ is also a distinguished vector bundle.

2.3.3 Local expressions and h - v -coordinates

In order to construct exact solutions in explicit forms by using geometric methods, it is important to use certain local coordinate and N-adapted constructions even if geometric models are (locally and/or globally) intrinsically formulated. Let us establish the necessary conventions: We use $x = \{x^i\}$ as local coordinates on a prime manifold M , when $i, j, \dots = 1, 2, \dots, n$. We use $y = \{y^a\}$ as local coordinates on a target manifold Q , when $a = n + 1, \dots, n + m$. On $J^r(M, Q)$, our local coordinates are x^i, y^a and the induced coordinates $v_{i_1 \dots i_p}^a$ are symmetric on the low indices $i_1, i_2, \dots, i_p = 1, \dots, n$, for $p = 1, \dots, r$. Working with nonholonomic jet spaces $\mathbf{J}^r(M, Q)$ for the same prime and target manifolds we use boldface induced coordinates $\mathbf{v}_{i_1 \dots i_p}^a$ which are not symmetric on i_1, i_2, \dots, i_p . We can consider corresponding coordinate systems with the same coordinate description for any $J^r Y, \mathbf{J}^r Y$ or $\mathbf{J}^r \mathbf{Y}$.

Let us introduce parameterizations for indices and coordinates of N-adapted maps $\mathbf{V} \rightarrow \mathbf{V}'$, when $\mathbf{u} = (\mathbf{x}, \mathbf{y}) = \{u^\alpha = (x^i, y^a)\}$ are local coordinates on \mathbf{V} and $\mathbf{u}' = (\mathbf{x}', \mathbf{y}') = \{u^{\alpha'} = (x^{i'}, y^{a'})\}$ are local coordinates on \mathbf{V}' . We write $\partial_{\tilde{\alpha}} f := \frac{\partial^{| \tilde{\alpha} |} f}{(\partial u^1)^{\alpha_1} \dots (\partial u^{n+m})^{\alpha_{n+m}}}$ for the partial derivative of a function $f : \mathbf{U} \subset \mathbb{R}^{n+m} \rightarrow \mathbb{R}$, with a **multi-index** $\tilde{\alpha}$ of range $n + m$, which is a $(n + m)$ -tuple $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{n+m})$ of non-negative integers. For such nonholonomic spaces, we write $|\tilde{\alpha}| = \alpha_1 + \dots + \alpha_{n+m}$, with $\tilde{\alpha}! = \alpha_1! \alpha_2! \dots \alpha_{n+m}!$, $0! = 1$, and $u^{\tilde{\alpha}} = (u^1)^{\alpha_1} \dots (u^{n+m})^{\alpha_{n+m}}$ for $u = (x^1, x^2, \dots, x^n; y^{n+1}, \dots, y^{n+m}) \in \mathbb{R}^{n+m}$.

The local coordinate system is conventionally split on both nonholonomic manifolds. Two N-adapted maps ${}^1 \mathbf{f} : \mathbf{V} \rightarrow \mathbf{V}'$ and ${}^2 \mathbf{f} : \mathbf{V} \rightarrow \mathbf{V}'$ satisfy $\mathbf{j}_{\mathbf{u}}^r {}^1 \mathbf{f} = \mathbf{j}_{\mathbf{u}}^r {}^2 \mathbf{f}$ if and only if, for the curves $u^\alpha = \zeta^\alpha t$ with arbitrary ζ^α ,

$$\sum_{|\tilde{\alpha}|=k} (\partial_{\tilde{\alpha}} {}^1 \mathbf{f}^{\alpha'}(u)) \zeta^{\tilde{\alpha}} = \sum_{|\tilde{\alpha}|=k} (\partial_{\tilde{\alpha}} {}^2 \mathbf{f}^{\alpha'}(u)) \zeta^{\tilde{\alpha}}, \text{ for } k = 0, 1, \dots, r.$$

We define jet distinguished groups, d-groups, with elements $L_{n+m,n'+m'}^r$ identified with the r -th order Taylor expansions of the generating maps. These are $n + m$ -tuples of polynomials of degree r in $n' + m'$ variables without absolute term, with a polynomial representative of a r -jet which can be written in the form $\sum_{1 \leq |\vec{\alpha}| \leq r} \zeta_{\vec{\alpha}}^{\alpha'} u^{\vec{\alpha}}$.

$L_{n+m,n'+m'}^r$ in a numerical space of the variables $\zeta_{\vec{\alpha}}^{\alpha'}$. A standard combinatoric calculus gives

$$\dim L_{n+m,n'+m'}^r = (n' + m') \left[\binom{n + m + r}{n + m} - 1 \right].$$

In explicit form, the coordinates on $L_{n+m,n'+m'}^r$ are denoted by $\zeta_{\vec{\alpha}}^{\alpha'}, \zeta_{\vec{\alpha}\vec{\beta}}^{\alpha'}, \dots, \zeta_{\vec{\alpha}_1 \dots \vec{\alpha}_r}^{\alpha'}$ which are symmetric in all subscripts if such values are taken in natural coordinate frames. The projection $\pi_s^r : L_{m,n}^r \rightarrow L_{m,n}^s$ consists in suppressing all terms of degree $> s$.

The set of all invertible elements of $L_{n+m,n'+m'}^r$ with the jet composition is a Lie d-group G_{n+m}^r called the r -th differential d-group of the r -th jet d-group in dimension $n + m$. For $r = 1$, the group G_{n+m}^1 is identified with a nonholonomic group decomposition $GL(n + m, \mathbb{R}) \rightarrow GL(n, \mathbb{R}) \oplus GL(m, \mathbb{R})$ corresponding to a horizontal (h) and vertical (v) splitting (1).

In this work, we study nonholonomic jet prolongations of the geometric objects from section 2.2 in $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ -framework with local coordinates

$$u^{\alpha_s} = (x^i, y^a, \zeta_{\vec{\alpha}_1 \dots \vec{\alpha}_r}^{\alpha'}) = (x^i, y^a, \zeta^{\alpha_s}). \quad (6)$$

We use the label s in order to perform a conventional splitting of dimensions, $\dim {}^sV = 4 + 2s = 2 + 2 + \dots + 2 \geq 4; s \geq 0$ for conventional finite dimensional (pseudo) Riemannian space sV . The jet coordinates $v_{\vec{\alpha}_1 \dots \vec{\alpha}_r}^{\alpha'}$ are re-grouped in oriented two shells⁸ which allows us to apply the AFDM and to construct exact solutions for generalized Einstein equations and metrics ${}^s\mathbf{g}$ with arbitrary signatures $(\pm 1, \pm 1, \pm 1, \dots \pm 1)$. Such shells are determined by nonholonomic data which transforms into $\zeta_{\vec{\alpha}_1 \dots \vec{\alpha}_r}^{\alpha'}$ with symmetric lower indices if the constructions are performed in coordinate bases. Let us establish conventions on (abstract) indices and coordinates $u^{\alpha_s} = (x^{i_s}, y^{a_s})$, for $s = 0, 1, 2, 3, \dots$ labellings of the oriented number of two dimensional, 2-d, "shells" added to a 4-d spacetime. For $s = 0$ (in a conventional form), we write $u^{\alpha} = (x^i, y^a)$ and consider the following local systems of coordinates:

$$\begin{aligned} s &= 1 : u^{\alpha_1} = (x^{\alpha} = u^{\alpha}, v^{a_1}) = (x^i, y^a, \zeta^{a_1}), \\ s &= 2 : u^{\alpha_2} = (x^{\alpha_1} = u^{\alpha_1}, v^{a_2}) = (x^i, y^a, \zeta^{a_1}, \zeta^{a_2}), \\ s &= 3 : u^{\alpha_3} = (x^{\alpha_2} = u^{\alpha_2}, v^{a_3}) = (x^i, y^a, \zeta^{a_1}, \zeta^{a_2}, \zeta^{a_3}), \dots \end{aligned} \quad (7)$$

for $i, j, \dots = 1, 2; a, b, \dots = 3, 4; a_1, b_1, \dots = 5, 6; a_2, b_2, \dots = 7, 8; a_3, b_3, \dots = 9, 10, \dots$ and $i_1, j_1, \dots = 1, 2, 3, 4; i_2, j_2, \dots = 1, 2, 3, 4, 5, 6; i_3, j_3, \dots = 1, 2, 3, 4, 5, 6, 7, 8; \dots$. In compact notation, we write $u = (x, y); {}^1u = (u, {}^1\zeta) = (x, y, {}^1\zeta), {}^2u = ({}^1u, {}^2\zeta) = (x, y, {}^1\zeta, {}^2\zeta), \dots$

We underline the indices in order to emphasize that certain values are with respect to local coordinate bases. The transformations between local frames, e_{α_s} , and coordinate frames, $\partial_{\underline{\alpha}_s} = \partial/\partial u^{\underline{\alpha}_s}$ on sV are written as $e_{\alpha_s} = e_{\alpha_s}^{\underline{\alpha}_s} ({}^s u) \partial/\partial u^{\underline{\alpha}_s}$. General parameterizations of coefficients $e_{\alpha_s}^{\underline{\alpha}_s}$ give nonholonomy relations $e_{\alpha_s} e_{\beta_s} - e_{\beta_s} e_{\alpha_s} = W_{\alpha_s \beta_s}^{\gamma_s} e_{\gamma_s}$. The nonholonomy coefficients $W_{\alpha_s \beta_s}^{\gamma_s} = W_{\beta_s \alpha_s}^{\gamma_s} (u)$ vanish for holonomic configurations. Using the condition $e^{\alpha_s} \rfloor e_{\beta_s} = \delta_{\beta_s}^{\alpha_s}$, where the 'hook' operator \rfloor corresponds to the inner derivative and $\delta_{\beta_s}^{\alpha_s}$ is the Kronecker symbol, we construct dual frames, $e^{\alpha_s} = e^{\alpha_s}_{\underline{\alpha}_s} ({}^s u) du^{\underline{\alpha}_s}$.

It is important to distinguish the partial derivatives on spacetime coordinates (for instance, $\partial_i = \partial/\partial x^i, \partial_a = \partial/\partial y^a$ and $\partial_{\alpha} = \partial/\partial u^{\alpha}$) and on r -jet variables, when $\tilde{\partial}_{a_s} = \partial/\zeta^{a_s}$ is used for a $2 + 2 + \dots$ conventional splitting of partial derivatives $\partial/\partial \zeta_{\vec{\alpha}_1 \dots \vec{\alpha}_r}^{\alpha'}$. In some sense, $\zeta_{\vec{\alpha}_1 \dots \vec{\alpha}_r}^{\alpha'}$ can be considered as extra dimension coordinates but with certain additional Lie d-group properties of G_{n+m}^r considered above.

2.4 Jet prolongation of Ricci soliton and Einstein equations

We can define canonical N-connection, frame, metric and distinguished metric structures on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ determined by prolongations of respective prime objects on \mathbf{V} , see Definition 2.3.

⁸In a similar form, we can split odd dimensions, for instance, $\dim V = 3 + 2 + \dots + 2$.

2.4.1 Shell parameterized N-connection and associated frame structures

A map $j^r f : M \rightarrow J^r(M, Q)$ is called a r -th jet prolongation of $f : M \rightarrow Q$. The set $J^r Q$ of all r -jets of the local sections of Y is called the r -th jet prolongation of Q and $J^r Q \subset J^r(M, Q)$ is a closed submanifold.

Theorem 2.1 *Any N-connection structure \mathbf{N} on \mathbf{V} determines a r -th jet prolongation of N-connection ${}^s\mathbf{N}$ on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ as the Whitney sum*

$${}^s\mathbf{N} : T {}^s\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \oplus {}^1v\mathbf{V} \oplus {}^2v\mathbf{V} \oplus \dots \oplus {}^sv\mathbf{V}, \quad (8)$$

for a conventional horizontal (h) and vertical (v) "shell by shell" splitting.

Proof. It is a natural construction when the coefficients of N-connection are defined by jet prolongations and parameterized as ${}^s\mathbf{N} = N_{i_s}^{a_s}({}^su)dx^{i_s} \otimes \partial/\partial\zeta^{a_s}$ on every chart on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$, i.e. for ${}^s\mathbf{V}$.

□ (end proof).

Using the coefficients of N-connection, we prove the following.

Corollary 2.1 *r -th jet prolongations induce on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ a system of N-elongated bases/ partial derivatives, $\mathbf{e}_{\nu_s} = (\mathbf{e}_{i_s}, \mathbf{e}_{a_s})$, and cobases, N-adapted differentials, $\mathbf{e}^{\mu_s} = (e^{i_s}, \mathbf{e}^{a_s})$.*

Proof. Taking (2) for \mathbf{V} , we prolongate on $s \geq 1$ shells,

$$\mathbf{e}_{i_s} = \frac{\partial}{\partial x^{i_s}} - N_{i_s}^{a_s} \partial_{a_s}, \quad \mathbf{e}_{a_s} = \partial_{a_s} = \frac{\partial}{\partial \zeta^{a_s}}, \quad (9)$$

$$e^{i_s} = dx^{i_s}, \quad \mathbf{e}^{a_s} = d\zeta^{a_s} + N_{i_s}^{a_s} dx^{i_s}. \quad (10)$$

□

The N-adapted operators (9) satisfy nonholonomy relations:

$$[\mathbf{e}_{\alpha_s}, \mathbf{e}_{\beta_s}] = \mathbf{e}_{\alpha_s} \mathbf{e}_{\beta_s} - \mathbf{e}_{\beta_s} \mathbf{e}_{\alpha_s} = W_{\alpha_s \beta_s}^{\gamma_s} \mathbf{e}_{\gamma_s}, \quad (11)$$

when $W_{i_s a_s}^{b_s} = \partial_{a_s} N_{i_s}^{b_s}$ and $W_{j_s i_s}^{a_s} = {}^J N_{i_s j_s}^{a_s}$, where the Neijenhuis tensor, i.e. the curvature of the r -th jet prolongation of N-connection, is ${}^J N_{i_s j_s}^{a_s} = \mathbf{e}_{j_s} (N_{i_s}^{a_s}) - \mathbf{e}_{i_s} (N_{j_s}^{a_s})$.

2.4.2 N-adapted shell prolongation of d-connections

On $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ with prolongation of geometric objects from \mathbf{V} , we define linear connection structures in N-adapted form in the following.

Theorem 2.2 –Definition: *There are distinguished connection, d-connection, structures, ${}^s\mathbf{D} = \{\mathbf{D}_{\alpha_s}\}$, with $\mathbf{D} = (hD; vD)$, ${}^1\mathbf{D} = ({}^1hD; {}^1vD)$, ..., ${}^{s-1}\mathbf{D} = ({}^{s-1}hD; {}^{s-1}vD)$, ${}^s\mathbf{D} = ({}^{s-1}hD; {}^svD)$, preserving under parallelism the N-connection splitting (8) and coefficients computed with respect to N-adapted bases (9) and (10).*

Proof. We shall use, for instance, the term **Theorem–Definition** when a geometric object is defined by an explicit construction which consists also of a proof of a Theorem. In this case, we prove the existence of a d-connection by considering N-adapted covariant derivatives

$$\begin{aligned} \mathbf{D}_{\alpha} &= (D_i; D_a), \mathbf{D}_{\alpha_1} = ({}^1D_{\alpha_1}; D_{a_1}), \mathbf{D}_{\alpha_2} = ({}^2D_{\alpha_2}; D_{a_2}), \dots, \mathbf{D}_{\alpha_s} = ({}^sD_{\alpha_{s-1}}; D_{a_s}), \text{ for} \\ hD &= (L_{jk}^i, L_{bk}^a), vD = (C_{jc}^i, C_{bc}^a), {}^1hD = (L_{\beta_1\gamma}^{\alpha}, L_{b_1\gamma}^{a_1}), {}^1vD = (C_{\beta_1c_1}^{\alpha}, C_{b_1c_1}^{a_1}), {}^2hD = (L_{\beta_1\gamma_1}^{\alpha_1}, L_{b_2\gamma_1}^{a_2}), \\ {}^2vD &= (C_{\beta_1c_2}^{\alpha_1}, C_{b_2c_2}^{a_2}), \dots, {}^shD = (L_{\beta_{s-1}\gamma_{s-1}}^{\alpha_{s-1}}, L_{b_s\gamma_{s-1}}^{a_s}), {}^svD = (C_{\beta_{s-1}c_s}^{\alpha_{s-1}}, C_{b_sc_s}^{a_s}), \end{aligned}$$

when the coefficients

$$\begin{aligned} \Gamma_{\beta\gamma}^{\alpha} &= (L_{jk}^i, L_{bk}^a; C_{jc}^i, C_{bc}^a), \Gamma_{\beta_1\gamma_1}^{\alpha_1} = (L_{\beta\gamma}^{\alpha}, L_{b_1\gamma}^{a_1}; C_{\beta c_1}^{\alpha}, C_{b_1c_1}^{a_1}), \\ \Gamma_{\beta_2\gamma_2}^{\alpha_2} &= (L_{\beta_1\gamma_1}^{\alpha_1}, L_{b_2\gamma_1}^{a_2}; C_{\beta_1c_2}^{\alpha_1}, C_{b_2c_2}^{a_2}), \dots, \Gamma_{\beta_s\gamma_s}^{\alpha_s} = (L_{\beta_{s-1}\gamma_{s-1}}^{\alpha_{s-1}}, L_{b_s\gamma_{s-1}}^{a_s}; C_{\beta_{s-1}c_s}^{\alpha_{s-1}}, C_{b_sc_s}^{a_s}) \end{aligned} \quad (12)$$

of such a d-connection ${}^s\mathbf{D} = \{\mathbf{D}_{\alpha_s}\}$ are computed in N-adapted form with respect to the frames (9)–(10) following equations $\mathbf{D}_{\alpha_s} \mathbf{e}_{\beta_s} = \Gamma_{\beta_s \gamma_s}^{\alpha_s} \mathbf{e}_{\gamma_s}$.

□

It is possible always to consider such frame transforms when all shell frames are N-adapted and ${}^1D_\alpha = \mathbf{D}_\alpha$, ${}^2D_{\alpha_1} = \mathbf{D}_{\alpha_1}, \dots, {}^sD_{\alpha_{s-1}} = \mathbf{D}_{\alpha_{s-1}}$.

Corollary 2.2 –Definition: *There are natural r -th jet prolongations of the torsion and curvature d-tensors (3) defined on a prime \mathbf{V} and elongated in N-adapted form on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ with prescribed shell splitting on ${}^s\mathbf{V}$,*

$${}^s\mathbf{T}(\mathbf{X}, \mathbf{Y}) := {}^s\mathbf{D}_\mathbf{X} \mathbf{Y} - {}^s\mathbf{D}_\mathbf{Y} \mathbf{X} + [\mathbf{X}, \mathbf{Y}] \quad \text{and} \quad (13)$$

$${}^s\mathbf{R}(\mathbf{X}, \mathbf{Y}) := {}^s\mathbf{D}_\mathbf{X} {}^s\mathbf{D}_\mathbf{Y} - {}^s\mathbf{D}_\mathbf{Y} {}^s\mathbf{D}_\mathbf{X} - {}^s\mathbf{D}_{[\mathbf{X}, \mathbf{Y}]}, \quad (14)$$

for any d-vectors $\mathbf{X}, \mathbf{Y} \subset T {}^s\mathbf{V}$.

Proof. To perform computations in N-adapted-shell form we consider a differential connection 1-form $\Gamma_{\beta_s}^{\alpha_s} = \Gamma_{\beta_s \gamma_s}^{\alpha_s} \mathbf{e}^{\gamma_s}$ and elaborate a differential form calculus with respect to skew symmetric tensor products of N-adapted frames (9)–(10). Respectively, the torsion $\mathcal{T}^{\alpha_s} = \{\mathbf{T}_{\beta_s \gamma_s}^{\alpha_s}\}$ and curvature $\mathcal{R}_{\beta_s}^{\alpha_s} = \{\mathbf{R}_{\beta_s \gamma_s \delta_s}^{\alpha_s}\}$ d-tensors of ${}^s\mathbf{D}$ are computed

$$\mathcal{T}^{\alpha_s} := {}^s\mathbf{D} \mathbf{e}^{\alpha_s} = d\mathbf{e}^{\alpha_s} + \Gamma_{\beta_s}^{\alpha_s} \wedge \mathbf{e}^{\beta_s}, \quad (15)$$

$$\mathcal{R}_{\beta_s}^{\alpha_s} := {}^s\mathbf{D} \Gamma_{\beta_s}^{\alpha_s} = d\Gamma_{\beta_s}^{\alpha_s} - \Gamma_{\beta_s}^{\gamma_s} \wedge \Gamma_{\gamma_s}^{\alpha_s} = \mathbf{R}_{\beta_s \gamma_s \delta_s}^{\alpha_s} \mathbf{e}^{\gamma_s} \wedge \mathbf{e}^{\delta_s}, \quad (16)$$

see Refs. [43, 45] for explicit calculations of coefficients $\mathbf{R}_{\beta_s \gamma_s \delta_s}^{\alpha_s}$ in higher dimensions. The formulae in the jet shell adapted coordinates (7) are very similar to those in N-adapted bases for extra dimensional (pseudo) Riemannian spaces. In standard r -jet coordinates (6) for $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$, $u^{\alpha_s} = (x^i, y^a, \zeta_{\alpha_1 \dots \alpha_r}^{\alpha'}$), additional contraction of up-down indices and symmetrization lead to very cumbersome coefficient formulae.

□

In Appendix A.1, we present two Theorems on computing N-adapted coefficients formulas for $\mathbf{T}_{\beta_s \gamma_s}^{\alpha_s}$ and $\mathbf{R}_{\beta_s \gamma_s \delta_s}^{\alpha_s}$.

2.4.3 Jet prolongation of d-metrics

On ${}^s\mathbf{V}$, a metric tensor can be written in the form

$${}^s\mathbf{g} = g_{\alpha_s \beta_s} e^{\alpha_s} \otimes e^{\beta_s} = g_{\underline{\alpha}_s \underline{\beta}_s} du^{\underline{\alpha}_s} \otimes du^{\underline{\beta}_s} = g_{\underline{\alpha}_s \underline{\beta}_s} du^{\underline{\alpha}_s} \otimes du^{\underline{\beta}_s} + g_{\underline{\alpha}_{s+1} \underline{\beta}_{s+1}} d\zeta^{\underline{\alpha}_{s+1}} \otimes d\zeta^{\underline{\beta}_{s+1}}, \quad s = 0, 1, 2, \dots, \quad (17)$$

where $du^{\underline{\alpha}} \in T^*\mathbf{V}$ and the indices are underlined in order to emphasize coordinate dual bases. The coefficients of such a metric are subject to frame transform rules, $g_{\alpha_s \beta_s} = e_{\alpha_s}^{\underline{\alpha}_s} e_{\beta_s}^{\underline{\beta}_s} g_{\underline{\alpha}_s \underline{\beta}_s}$, which can be respectively generalized for any tensor object. We can not preserve a $2 + 2 + 2 + \dots$ splitting of the dimensions under general frame/coordinate transforms.

Lemma 2.1 *Any metric structure ${}^s\mathbf{g} = \{\mathbf{g}_{\alpha_s \beta_s}\}$ on ${}^s\mathbf{V}$ can be written as a distinguished metric (d-metric)*

$$\begin{aligned} {}^s\mathbf{g} &= g_{i_s j_s}({}^s u) e^{i_s} \otimes e^{j_s} + g_{a_s b_s}({}^s u) \mathbf{e}^{a_s} \otimes \mathbf{e}^{b_s} \\ &= g_{ij}(x) e^i \otimes e^j + g_{ab}(u) \mathbf{e}^a \otimes \mathbf{e}^b + g_{a_1 b_1}({}^1 u) \mathbf{e}^{a_1} \otimes \mathbf{e}^{b_1} + \dots + g_{a_s b_s}({}^s u) \mathbf{e}^{a_s} \otimes \mathbf{e}^{b_s}. \end{aligned} \quad (18)$$

Proof. Using frame/coordinate transforms, we can always parameterize any metric (17) in such form:

$$g_{\alpha\beta}(u) = \begin{bmatrix} g_{ij} + h_{ab} N_i^a N_j^b & h_{ae} N_j^e \\ h_{be} N_i^e & h_{ab} \end{bmatrix},$$

on the prime manifold and, for r -prolongations,

$$\begin{aligned} \underline{g}_{\alpha_1\beta_1}({}^1\zeta) &= \begin{bmatrix} \underline{g}_{\alpha\beta} & h_{a_1e_1}N_{\beta_1}^{e_1} \\ h_{b_1e_1}N_{\alpha_1}^{e_1} & h_{a_1b_1} \end{bmatrix}, \quad \underline{g}_{\alpha_2\beta_2}({}^2\zeta) = \begin{bmatrix} \underline{g}_{\alpha_1\beta_1} & h_{a_2e_2}N_{\beta_1}^{e_2} \\ h_{b_2e_2}N_{\alpha_1}^{e_2} & h_{a_2b_2} \end{bmatrix}, \dots \\ \underline{g}_{\alpha_s\beta_s}({}^s\zeta) &= \begin{bmatrix} g_{i_sj_s} + h_{a_sb_s}N_{i_s}^{a_s}N_{j_s}^{b_s} & h_{a_se_s}N_{j_s}^{e_s} \\ h_{b_se_s}N_{i_s}^{e_s} & h_{a_sb_s} \end{bmatrix}. \end{aligned}$$

By re-grouping terms shell by shell with respect to the bases (10), we obtain (18).

□

In diverse dimensions, such parameterizations are similar to those introduced in Kaluza–Klein type theories when $\zeta^{a_s}, s \geq 1$, are considered as extra dimension coordinates with cylindrical compactification and $N_\alpha^{e_s}({}^su) \sim A_{a_s\alpha}^{e_s}(u)y^\alpha$ represent certain (non) Abelian gauge fields $A_{a_s\alpha}^{e_s}(u)$. Jet generalized gauge theories possess different symmetries than those with potentials taking values in the Lie group algebras, see Ref. [1] on unification of gravity with internal gauge interactions.

2.4.4 Canonical jet distortions and linear connections on jet bundles

For any r -jet prolongation (pseudo) Riemannian metric ${}^s\mathbf{g}$, we can construct in standard form the Levi–Civita connection (LC-connection), ${}^s\nabla = \{{}_1\Gamma_{\beta_s\gamma_s}^{\alpha_s}\}$. By definition such a connection is metric compatible, ${}^s\nabla({}^s\mathbf{g}) = 0$, and with zero torsion, ${}_1T^{\alpha_s} = 0$ (we use formulae (15) for ${}^s\mathbf{D} \rightarrow {}^s\nabla$). It should be emphasized that such a linear connection is not a d-connection because it does not preserve under general coordinate transforms a N-connection splitting (8).

Theorem 2.3 *There is a canonical distortion relation*

$${}^s\widehat{\mathbf{D}} = {}^s\nabla + {}^s\widehat{\mathbf{Z}}, \quad (19)$$

for a canonical d-connection ${}^s\widehat{\mathbf{D}}$ which is completely and uniquely defined by a (pseudo) Riemannian metric ${}^s\mathbf{g}$ (18) for a chosen nonholonomic distribution ${}^s\mathbf{N} = \{N_{i_s}^{a_s}\}$ when ${}^s\widehat{\mathbf{D}}({}^s\mathbf{g}) = 0$ and the horizontal and vertical torsions are zero, i.e. $h\widehat{\mathbf{T}} = \{\widehat{\mathbf{T}}_{jk}^i\} = 0$, $v\widehat{\mathbf{T}} = \{\widehat{\mathbf{T}}_{bc}^a\} = 0$, ${}_1v\widehat{\mathbf{T}} = \{\widehat{\mathbf{T}}_{b_1c_1}^{a_1}\} = 0, \dots$, ${}_sv\widehat{\mathbf{T}} = \{\widehat{\mathbf{T}}_{b_sc_s}^{a_s}\} = 0$; the distorting tensor ${}^s\widehat{\mathbf{Z}} = \{\widehat{\mathbf{Z}}_{\beta_s\gamma_s}^{\alpha_s}\}$ is uniquely defined by the same data $({}^s\mathbf{g}, {}^s\mathbf{N})$.

Proof. We sketch a proof in appendix A.2.

□

The N-adapted coefficients of the distortion d-tensor $\widehat{\mathbf{Z}}_{\beta_s\gamma_s}^{\alpha_s}$ are algebraic combinations of $\widehat{T}_{\beta_s\gamma_s}^{\alpha_s}$ and vanish for zero torsion. The nonholonomic variables $({}^s\mathbf{g} \text{ (18)}, {}^s\mathbf{N}, {}^s\widehat{\mathbf{D}})$ are equivalent to the standard (pseudo) Riemannian ones $({}^s\mathbf{g} \text{ (17)}, {}^s\nabla)$. For instance, GR in 4-d can be formulated equivalently using the connection ∇ and/or $\widehat{\mathbf{D}}$ if the distortion relation (5) is used, see details in [40, 43, 45]. The r -jet prolongations give distortions (19). We consider nonholonomic jet deformations of a 4-d (pseudo) Riemannian space to a $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ with a canonical nonzero d-torsion. In such cases, we are able to decouple modified Einstein equations and construct integral varieties with jet variables. At the end, we can impose additional nonholonomic constraints and fix the jet coordinates in order to generate exact solutions of Ricci soliton/ Einstein equations in 4-d, or higher dimensions, with r -jet symmetries.

Here we note that ${}^s\nabla$ and ${}^s\widehat{\mathbf{D}}$ are not tensor objects. ${}^s\widehat{\mathbf{D}}$ is a d-connection and such linear connections are subject to different rules with respect to coordinate transformations. It is possible to consider frame transformations with certain ${}^s\mathbf{N} = \{N_{i_s}^{a_s}\}$ when the conditions ${}_1\Gamma_{\alpha_s\beta_s}^{\gamma_s} = \widehat{\Gamma}_{\alpha_s\beta_s}^{\gamma_s}$ are satisfied with respect to some N-adapted frames (9)–(10). In general, ${}^s\nabla \neq {}^s\widehat{\mathbf{D}}$ and the corresponding curvature tensors ${}_1R_{\beta_s\gamma_s\delta_s}^{\alpha_s} \neq \widehat{\mathbf{R}}_{\beta_s\gamma_s\delta_s}^{\alpha_s}$ are different, but the Ricci tensor components may coincide for certain classes of nonholonomic constraints.

2.4.5 Prolongation of Ricci soliton and Einstein equations on nonholonomic jet configurations

In this section, we introduce important geometric and physical equations in nonholonomic variables on \mathbf{V} and consider generalizations on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$.

Definition 2.7 *The geometric data $(\mathbf{g}, \mathbf{N}, \mathbf{D}; \mathbf{V})$ defines a gradient nonholonomic Ricci soliton if there exists a smooth potential function $\kappa(x, y)$ such that*

$$\widehat{\mathbf{R}}_{\beta\gamma} + \widehat{\mathbf{D}}_{\beta}\widehat{\mathbf{D}}_{\gamma}\kappa = \lambda \mathbf{g}_{\beta\gamma}. \quad (20)$$

There are three types of such Ricci solitons determined by a constant λ : steady ones for $\lambda = 0$; shrinking ones for $\lambda > 0$; and expanding ones for $\lambda < 0$.

The above classification is determined by the Levi–Civita, LC, limits when shrinking solutions help us to understand the asymptotic behaviour of the ancient (old) solutions of the Ricci flow theory [17, 18, 33]. By generalizing and adapting the constructions to N–connection structures, one can describe geometric flows with nonholonomic constraints [41, 42]. Here, we omit a study of geometric analysis issues and generalized Ricci flow models and restrict our research to nonholonomic r –jet prolongations of equations and important classes of solutions.

The N–adapted coefficients of the Ricci d–tensor $Ric = \{\mathbf{R}_{\alpha_s\beta_s} := \mathbf{R}_{\alpha_s\beta_s\tau_s}^{\tau_s}\}$ of a d–connection ${}^s\mathbf{D}$ in $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ are computed from the curvature tensor (16),

$$\mathbf{R}_{\alpha_s\beta_s} = \{R_{i_s j_s} := R_{i_s j_s k_s}^{k_s}, \quad R_{i_1 a_1} := -R_{i_1 k_1 a_1}^{k_1}, \dots, \quad R_{a_s i_s} := R_{a_s i_s b_s}^{b_s}\}. \quad (21)$$

Using the inverse matrix of ${}^s\mathbf{g}$ (18), we compute the scalar curvature of ${}^s\mathbf{D}$,

$${}^sR := \mathbf{g}^{\alpha_s\beta_s}\mathbf{R}_{\alpha_s\beta_s} = g^{i_s j_s}R_{i_s j_s} + h^{a_s b_s}R_{a_s b_s} = R + S + {}^1S + \dots + {}^sS, \quad (22)$$

with respective to the horizontal (h) and vertical (v) components of the scalar curvature, $R = g^{ij}R_{ij}$, $S = h^{ab}R_{ab}$, ${}^1S = h^{a_1 b_1}R_{a_1 b_1}, \dots$, ${}^sS = h^{a_s b_s}R_{a_s b_s}$.

The Einstein d–tensor ${}^s\mathbf{Einst} = \{{}^s\mathbf{E}_{\alpha_s\beta_s}\}$ for any nonholonomic r –jet data $({}^s\mathbf{g}, {}^s\mathbf{N}, {}^s\mathbf{D})$ is determined in standard forms as,

$${}^s\mathbf{E}_{\alpha_s\beta_s} := {}^s\mathbf{R}_{\alpha_s\beta_s} - \frac{1}{2}\mathbf{g}_{\alpha_s\beta_s} {}^sR. \quad (23)$$

Such nonholonomic jet prolongations of a prime Einstein tensor are not symmetric, and the d–tensor ${}^s\mathbf{R}_{\alpha_s\beta_s}$ is not symmetric for a general ${}^s\mathbf{D}$ and ${}^s\mathbf{D}({}^s\mathbf{Einst}) \neq 0$. For a canonical ${}^s\widehat{\mathbf{D}}$, we can always compute ${}^s\widehat{\mathbf{D}}({}^s\widehat{\mathbf{Einst}})$ as a unique distortion relation determined by (19).

Proposition 2.1 *For a N–connection splitting (8) on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$, the Levi–Civita and canonical d–connection with prolongations are defined by the conditions*

$${}^s\mathbf{g} \rightarrow \begin{cases} {}^s\nabla : & {}^s\nabla {}^s\mathbf{g} = \mathbf{0}; \quad {}^s\nabla \mathbf{T} = \mathbf{0}, & \text{the Levi–Civita connection;} \\ {}^s\widehat{\mathbf{D}} : & {}^s\widehat{\mathbf{D}} {}^s\mathbf{g} = \mathbf{0}; \quad h\widehat{\mathbf{T}} = \mathbf{0}, \quad {}^1v\widehat{\mathbf{T}} = \mathbf{0}, \dots, \quad {}^sv\widehat{\mathbf{T}} = \mathbf{0}. & \text{the canonical d–connection.} \end{cases}$$

Proof. It is a trivial N–adapted construction with nonholonomic r –jet prolongations of (4).

□

Einstein equations for a metric ${}^s\mathbf{g}$ are natural prolongations that can be formulated in standard form using the LC–connection ${}^s\nabla$. By computing the corresponding Ricci tensor, ${}_{\mathbf{I}}R_{\alpha_s\beta_s}$, curvature scalar, ${}_{\mathbf{I}}R$, and Einstein tensor, ${}_{\mathbf{I}}E_{\alpha_s\beta_s}$, we arrive at

$${}_{\mathbf{I}}E_{\alpha_s\beta_s} := {}_{\mathbf{I}}R_{\alpha_s\beta_s} - \frac{1}{2}g_{\alpha_s\beta_s} {}_{\mathbf{I}}R = \varkappa {}_{\mathbf{I}}T_{\alpha_s\beta_s}, \quad (24)$$

where \varkappa is the gravitational constant and ${}_{\mathbf{I}}T_{\alpha_s\beta_s}$ is the stress–energy tensor for matter fields. In 4-d, there are well-defined geometric/variational and physically motivated procedures for constructing ${}_{\mathbf{I}}T_{\alpha_s\beta_s}$. Such values can

be similarly (at least geometrically) re-defined with respect to N-adapted frames using distortion relations (19) and introducing extra dimensions.

The gravitational field equations (24) can be rewritten equivalently in N-adapted form for the canonical d-connection ${}^s\widehat{\mathbf{D}}$, as

$${}^s\widehat{\mathbf{R}}_{\beta_s\delta_s} - \frac{1}{2}\mathbf{g}_{\beta_s\delta_s} {}^sR = \Upsilon_{\beta_s\delta_s}, \quad (25)$$

$$\widehat{L}_{asjs}^{cs} = e_{as}(N_{js}^{cs}), \quad \widehat{C}_{jsbs}^{is} = 0, \quad {}^N\widehat{J}_{jsis}^{as} = 0. \quad (26)$$

The sources $\Upsilon_{\beta_s\delta_s}$ are constructed as in GR but with nonholonomic jet deformations for the formal extra dimensions, when $\Upsilon_{\beta_s\delta_s} \rightarrow \varkappa T_{\beta_s\delta_s}$ for ${}^s\widehat{\mathbf{D}} \rightarrow {}^s\nabla$. The solutions of (25) contain nonholonomically induced torsion (15).

If the conditions (26) are satisfied, the d-torsion coefficients (A.4) are zero and we get the LC-connection, i.e. it is possible to "extract" solutions like as for the standard Einstein equations. The decoupling property can be proved in explicit form by working with ${}^s\widehat{\mathbf{D}}$ and nonholonomic torsion configurations. Having constructed certain classes of solutions in explicit form, with nonholonomically induced torsions and depending on various sets of integration and generating functions and parameters, we can "extract" the solutions for ${}^s\nabla$ by imposing at the end additional constraints that give zero torsion.

Using natural prolongations from \mathbf{V} on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$, we prove

Theorem 2.4 *In nonholonomic N-adapted r-jet variables, the gradient canonical Ricci jet-solitons are defined by equations*

$$\widehat{\mathbf{R}}_{\beta_s\gamma_s} + \widehat{\mathbf{D}}_{\beta_s}\widehat{\mathbf{D}}_{\gamma_s}\kappa = \lambda\mathbf{g}_{\beta_s\gamma_s}. \quad (27)$$

Our first goal is to elaborate a geometric method for decoupling the equations (20) and (27). This is possible for certain classes of nonholonomic constraints when the systems of nonlinear PDE (25) is supplemented with additional zero torsion conditions (26). We are able to find general classes of solutions when κ is parameterized so as to satisfy

$$\widehat{\mathbf{R}}_{ij} = {}^h\Upsilon(x^k)\mathbf{g}_{ij}, \quad (28)$$

$$\widehat{\mathbf{R}}_{ab} = {}^v\Upsilon(x^k, y^a)\mathbf{g}_{ab}, \quad (29)$$

$$\widehat{\mathbf{R}}_{\beta\gamma} = 0, \text{ for } \beta \neq \gamma, \quad (30)$$

$$\widehat{\mathbf{R}}_{a_1b_1} = {}^1v\Upsilon(x^k, y^a, \zeta^{a_1})\mathbf{g}_{a_1b_1}, \quad (31)$$

$$\widehat{\mathbf{R}}_{\beta_1\gamma_1} = 0, \text{ for } \beta_1 \neq \gamma_1, \quad (32)$$

...

$$\widehat{\mathbf{R}}_{a_sb_s} = {}^sv\Upsilon(x^k, y^a, \zeta^{a_1}, \dots, \zeta^{a_s})\mathbf{g}_{a_sb_s}, \quad (33)$$

$$\widehat{\mathbf{R}}_{\beta_s\gamma_s} = 0, \text{ for } \beta_s \neq \gamma_s, \quad (34)$$

with respect to N-adapted frames (9) and (10). The effective source (anisotropically polarized cosmological constant)

$$\Upsilon_\beta^\alpha = \text{diag}[\Upsilon_1^1 = \Upsilon_2^2 = {}^h\Upsilon, \Upsilon_3^3 = \Upsilon_4^4 = {}^v\Upsilon, \Upsilon_5^5 = \Upsilon_6^6 = {}^1v\Upsilon, \dots, \Upsilon_{2s-1}^{2s-1} = \Upsilon_{2s}^{2s} = {}^sv\Upsilon], \quad (35)$$

is parameterized accordingly that allows the integration of the differential equations in explicit forms. We shall prove that the general classes of solutions $\mathbf{g}_{\alpha\beta}(u)$ depend generically on all spacetime coordinates via the corresponding generating and integration functions, and various integrations constants. The solutions will represent an explicit application of the geometry of nonholonomic distributions and generalized connections in

mathematical relativity, modified gravity theories and the theory of physically important nonlinear systems of PDEs.

The second goal is to find explicit solutions for the Levi-Civita (LC-configurations) (26) , i.e. for the additional constraints when

$${}^s\widehat{\mathbf{T}} = 0, \quad (36)$$

(this formula follows from (13)), when for some classes of solutions (${}^s\mathbf{g}$, ${}^s\mathbf{N}$, ${}^s\widehat{\mathbf{D}}$) of (28)–(34) we can extract certain subvarieties of solutions (${}^s\check{\mathbf{g}}$, ${}^s\check{\mathbf{N}}$, ${}^s\nabla = {}^s\check{\mathbf{D}} = {}^s\mathbf{D}|_{{}^s\check{\mathbf{T}} \rightarrow \mathbf{0}}$) for zero torsion, after re-scaling the generating functions and sources, ${}^s\mathbf{g} \rightarrow {}^s\check{\mathbf{g}}$, ${}^s\mathbf{N} \rightarrow {}^s\check{\mathbf{N}}$ and ${}^h\Upsilon(x^k), \dots, {}^v\Upsilon(x^k) \rightarrow \lambda = \text{const.}$ With this procedure, we formulate a geometric method of constructing exact solutions of the Einstein equations (24) with re-defined source and N-adapted frame structures when

$$R_{\alpha_s\beta_s}[{}^s\nabla] = \lambda\check{\mathbf{g}}_{\alpha_s\beta_s} \quad (37)$$

on nonholonomic manifolds/bundles and r -jet prolongations. The metrics contain generic off-diagonal elements (i.e. can not be diagonalized via coordinate transforms), which may depend on all spacetime coordinates and can be prescribed (via generating/integrating functions and constants) to satisfy various necessary types of symmetry, boundary, Cauchy, and topological conditions, with possible singularities and horizons. We note that the system (28)–(34) together with LC-conditions (36) is equivalent to (37). Both such systems of PDEs are nonlinear and parameter dependent. So, it is important at what stage certain nonholonomic constraints and ansatz conditions for frames, metrics and connections are imposed i.e., at the end, when some solutions for ${}^s\widehat{\mathbf{D}}$ have been found, or at the beginning, when ${}^s\widehat{\mathbf{D}} \rightarrow {}^s\nabla$. We can not decouple such systems of equations in a general form if we work from the very beginning with the Levi-Civita connection ${}^s\nabla$.

The third goal is to lay down certain geometric conditions on when a general (pseudo) Riemannian manifold (\mathbf{g}, \mathbf{V}) (the metric \mathbf{g} may or not be a solution of any (modified) Einstein or Ricci soliton equations) can be non-holonomically deformed via the corresponding nonholonomic jet maps with generalized connection structures into certain geometrically/ physically important classes of solutions of systems of the type (28)–(36), or (37).

nonholonomic *Jet*

In such cases $(\mathbf{g}, \mathbf{V}) \dashrightarrow (\check{\mathbf{g}}, \check{\mathbf{N}}, \check{\mathbf{D}}, \check{\mathbf{V}})$, when the target space $\check{\mathbf{V}}$ and the fundamental

geometric structures $(\check{\mathbf{g}}, \check{\mathbf{N}}, \check{\mathbf{D}})$ are constructed with nonholonomic jet transformations and (*Jet*) are solutions of certain (modified/generalized) Einstein or Ricci soliton equations that depend on generalized jet parameters and corresponding jet symmetries. We note, that for such geometric and physical models the jet variables are prescribed certain constant values (we suppress the left label s).

3 Decoupling and Integration of Jet Prolongation of Einstein Equations

In this section, we prove that the system of nonlinear PDEs (28)–(34) with possible constraints (36) giving (37), can be decoupled in very general forms with respect to N-adapted frames with two dimensional shell parameterizations of jet variables. We show how such decoupled systems can be integrated in general forms for vacuum and non-vacuum solutions in (modified) gravity and Ricci soliton theories.

3.1 Off-diagonal metrics for r -jet configurations with one Killing symmetry

We study nonholonomic jet deformations of the "primary" geometric/physical data into "target" data,

$$[\text{primary}]({}^o\mathbf{g}, {}^o\mathbf{N}, {}^o\widehat{\mathbf{D}}) \rightarrow [\text{target}]({}^s_\eta\mathbf{g} = {}^s\mathbf{g}, {}^s_\eta\mathbf{N} = {}^s\mathbf{N}, {}^s_\eta\widehat{\mathbf{D}} = {}^s\widehat{\mathbf{D}}).$$

The values labeled by "o" define exact solutions in a Ricci soliton or gravity theory. The metrics with left label " η " define a solution of modified gravitational field equations (28)–(34). The prime ansatz is written

$$\begin{aligned} {}^o\mathbf{g} &= \dot{g}_i(x^k)dx^i \otimes dx^i + \dot{h}_a(x^k, y^4)\dot{\mathbf{e}}^a \otimes \dot{\mathbf{e}}^b + \epsilon_{a_1} dy^{a_1} \otimes dy^{a_1} + \dots + \epsilon_{a_s} dy^{a_s} \otimes dy^{a_s}, \\ \dot{\mathbf{e}}^a &= dy^a + \dot{N}_i^a(x^k, y^4)dx^i, \text{ with } \dot{N}_i^3 = \dot{n}_i, \dot{N}_i^4 = \dot{w}_i, \end{aligned} \quad (38)$$

for $\epsilon_{a_s} = \pm 1$ depending on the signature of extra dimensions and $(\hat{g}_i, \hat{h}_a; \hat{N}_i^a)$ defining, for instance, the Kerr black hole solution trivially imbedded into a $4 + 2s$ jet prolongation of spacetime. We express the N-adapted coefficients of a target ansatz (18) as

$$g_{\alpha_s} = \eta_{\alpha_s}(u^{\beta_s})\hat{g}_{\alpha_s}; N_{i_s}^{a_s} = \eta N_{i_s}^{a_s}(u^{\beta_{s-1}}, y^{4+2s}); n_i = \eta_i^3 \hat{n}_i, w_i = \eta_i^4 \hat{w}_i, \text{ not summation over } i; \quad (39)$$

with the so-called gravitational "polarization" functions and extra dimensional N-coefficients, $\eta_{\alpha_s}, \eta_i^a$ and $\eta N_{i_s}^{a_s}$. To be able to study certain limits $(\overset{s}{\eta}\mathbf{g}, \overset{s}{\eta}\mathbf{N}, \overset{s}{\eta}\hat{\mathbf{D}}) \rightarrow (\overset{s}{\circ}\mathbf{g}, \overset{s}{\circ}\mathbf{N}, \overset{s}{\circ}\hat{\mathbf{D}})$, for $\varepsilon \rightarrow 0$, depending on a small parameter $\varepsilon, 0 \leq \varepsilon \ll 1$, we introduce "small" polarizations of the type $\eta = 1 + \varepsilon\chi(u...)$ and $\eta N_{i_s}^{a_s} = \varepsilon n_{i_s}^{a_s}(u...)$.

The decoupling property of modified Einstein equations can be proven in the simplest form for certain ansatz with at least one Killing symmetry on a spacetime coordinate and certain parameterizations of nonholonomic r -jet prolongations. We consider target metrics of type (18) parameterized in the form

$$\begin{aligned} \overset{s}{\omega}\mathbf{g} &= g_i(x^k)dx^i \otimes dx^i + h_a(x^k, y^4)\mathbf{e}^a \otimes \mathbf{e}^b + \\ & h_{a_1}(u^\alpha, \zeta^6) \mathbf{e}^{a_1} \otimes \mathbf{e}^{a_1} + h_{a_2}(u^{\alpha_1}, \zeta^8) \mathbf{e}^{a_2} \otimes \mathbf{e}^{b_2} + \dots + h_{a_s}(u^{\alpha_{s-1}}, \zeta^{a_s})\mathbf{e}^{a_s} \otimes \mathbf{e}^{a_s}, \\ \text{where } \mathbf{e}^a &= dy^a + N_i^a dx^i, \text{ for } N_i^3 = n_i(x^k, y^4), N_i^4 = w_i(x^k, y^4); \\ \mathbf{e}^{a_1} &= d\zeta^{a_1} + N_{\alpha_1}^{a_1} du^\alpha, \text{ for } N_{\alpha_1}^5 = {}^1n_\alpha(u^\beta, \zeta^6), N_{\alpha_1}^6 = {}^1w_\alpha(u^\beta, \zeta^6); \\ \mathbf{e}^{a_2} &= d\zeta^{a_2} + N_{\alpha_1}^{a_2} du^{\alpha_1}, \text{ for } N_{\alpha_1}^7 = {}^2n_{\alpha_1}(u^{\beta_1}, \zeta^8), N_{\alpha_1}^8 = {}^2w_{\alpha_1}(u^{\beta_1}, \zeta^8); \\ &\dots \\ \mathbf{e}^{a_s} &= d\zeta^{a_s} + N_{\alpha_{s-1}}^{a_s} du^{\alpha_{s-1}}, \text{ for } N_{\alpha_{s-1}}^{4+2s-1} = {}^s n_{\alpha_1}(u^{\beta_{s-1}}, \zeta^{4+2s}), N_{\alpha_1}^{4+2s} = {}^s w_{\alpha_1}(u^{\beta_{s-1}}, \zeta^{4+2s}). \end{aligned} \quad (40)$$

Such ansatz also contains a jet Killing vector $\partial/\partial\zeta^{s-1}$ because the jet coordinate ζ^{s-1} is not contained in the coefficients of such metrics.

3.2 Decoupling in nonholonomic r -jet shell variables

Let us consider an ansatz (40) with $g_i(x^k) = \epsilon_i e^{\psi(x^k)}$, where $\epsilon_i = \pm 1$, and the γ, α, β -coefficients are defined by respective generating functions $\phi, {}^s\phi$ following the formulae

$$\gamma := \partial_4(\ln|h_3|^{3/2}/|h_4|), \quad \alpha_i = (\partial_i\phi)(\partial_4 h_3)/2h_3, \quad \beta = (\partial_4\phi)(\partial_4 h_3)/2h_3, \quad (41)$$

$$\text{for generating function } \phi = \ln \left| \partial_4 h_3 / \sqrt{|h_3 h_4|} \right|, \quad (42)$$

$${}^1\gamma := \partial_6(\ln|h_5|^{3/2}/|h_6|), \quad {}^1\alpha_\tau = (\partial_\tau {}^1\phi)(\partial_6 h_5)/2h_5, \quad {}^1\beta = (\partial_\tau {}^1\phi)(\partial_6 h_5)/2h_5, \quad (43)$$

$$\text{for } r\text{-jet generating function } {}^1\phi = \ln \left| (\partial_6 h_5) / \sqrt{|h_5 h_6|} \right|, \quad (44)$$

$${}^2\gamma := \partial_8(\ln|h_7|^{3/2}/|h_8|), \quad {}^2\alpha_{\tau_1} = (\partial_{\tau_1} {}^2\phi)(\partial_8 h_7)/2h_7, \quad {}^2\beta = (\partial_{\tau_1} {}^2\phi)(\partial_8 h_7)/2h_7,$$

$$\text{for } r\text{-jet generating function } {}^s\phi = \ln \left| \partial_{2s} h_{2s-1} / \sqrt{|h_{2s-1} h_{2s}|} \right|,$$

....,

with nonzero $\partial_4\phi, \partial_4 h_a, \partial_6 {}^1\phi, \partial_6 h_{a_1}, \partial_{2s} {}^2\phi, \partial_{2s} h_{a_2}$.

We assume that via the N-adapted frame transformations the sources $\mathbf{T}_{\beta_s \delta_s}$ (35) in the equations (28), (29), (31) and (33) can be parameterized in the form

$$\begin{aligned} \mathbf{T}_1^1 &= \mathbf{T}_2^2 = {}^v\Lambda(x^k, y^4) + {}^v_1\Lambda(u^\beta, \zeta^6) + {}^v_2\Lambda(u^{\beta_1}, \zeta^8), \mathbf{T}_3^3 = \mathbf{T}_4^4 = \Lambda(x^k) + {}^v_1\Lambda(u^\beta, \zeta^6) + {}^v_2\Lambda(u^{\beta_1}, \zeta^8), \\ \mathbf{T}_5^5 &= \mathbf{T}_6^6 = \Lambda(x^k) + {}^v\Lambda(x^k, y^4) + {}^v_2\Lambda(u^{\beta_1}, \zeta^8), \mathbf{T}_7^7 = \mathbf{T}_8^8 = \Lambda(x^k) + {}^v\Lambda(x^k, y^4) + {}^v_1\Lambda(u^\beta, \zeta^6). \end{aligned} \quad (45)$$

Such parameterizations are very general for (effective) $\mathbf{T}_{\beta_s \delta_s}$ with arbitrary contributions from Ricci soliton or (modified) gravity and matter fields and further r -jet generalizations when the N-adapted coefficients are modeled in certain systems of references by "polarized" cosmological constants $\Lambda(x^k), {}^v\Lambda(x^k, y^4), {}^v_1\Lambda(u^\beta, \zeta^6), {}^v_2\Lambda(u^{\beta_1}, \zeta^8)$ etc. For certain models of gravity in extra dimensions, we consider, for simplicity, $\Lambda = {}^v\Lambda = {}^v_1\Lambda = {}^v_2\Lambda = \text{const}$. Such effective sources can always be introduced by re-defining the generating functions (see below) for non-vacuum configurations.

Theorem 3.1 *For a general off-diagonal ansatz (40) with Killing symmetry ∂_3 and N -adapted parameterizations for generating functions (41)–(44) and sources (45), the system of modified Einstein equations (see N -adapted equations (A.5)–(A.12)) decouple in the following form:*

For a nonholonomic 2+2 spacetime splitting,

$$\epsilon_1 \partial_{11} \psi + \epsilon_2 \partial_{22} \psi = 2\Lambda(x^k), \quad (46)$$

$$(\partial_4 \phi)(\partial_4 h_3) = 2h_3 h_4 {}^v \Lambda(x^k, y^4), \quad (47)$$

$$\partial_{44} n_i + \gamma \partial_4 n_i = 0, \quad (48)$$

$$\beta w_i - \alpha_i = 0; \quad (49)$$

and, on nonholonomic r -jet variables,

$$(\tilde{\partial}_6 {}^1 \phi)(\tilde{\partial}_6 h_5) = 2h_5 h_6 {}^v \Lambda(u^\beta, \zeta^6), \quad (50)$$

$$\tilde{\partial}_{66}^2 {}^1 n_\tau + {}^1 \gamma \tilde{\partial}_6 {}^1 n_\tau = 0, \quad (51)$$

$${}^1 \beta {}^1 w_\tau - {}^1 \alpha_\tau = 0, \quad (52)$$

...

$$(\tilde{\partial}_{2s} {}^s \phi)(\tilde{\partial}_{2s} h_{2s-1}) = 2h_{2s-1} h_{2s} {}^v \Lambda(u^{\beta_{s-1}}, \zeta^{2s}),$$

$$\tilde{\partial}_{2s}^2 {}^{2s} n_{\tau_1} + {}^{2s} \gamma \tilde{\partial}_{2s} {}^{2s} n_{\tau_1} = 0,$$

$${}^{2s} \beta {}^{2s} w_{\tau_1} - {}^{2s} \alpha_{\tau_1} = 0, \quad (53)$$

....

Proof. It is a tedious technical proof following an explicit calculation of nontrivial components of the Ricci d-tensor for the mentioned ansatz and parameterizations of (effective) sources and parametrization functions (see Appendix A.3). Such equations are straightforward consequences of the symmetries of the Einstein and the Ricci d-tensors (A.13) for the canonical d-connection ${}^s \hat{\mathbf{D}}$.

□

Let us explain in brief the decoupling property for the nonholonomic 4-d and r -jet equations (46)–(53):

1. The equation (46) is just a 2-d Laplace, or d’Alambert equation (depending on the prescribed signature for $\epsilon_i = \pm 1$) with solutions determined by any source $\Lambda(x^k)$.
2. The equation (47) contains only the partial derivative ∂_4 and constitutes, together with the algebraic formula for the coefficient (42), a system of two equations for four functions: $h_3(x^i, y^4)$, $h_4(x^i, y^4)$ and $\phi(x^i, y^4)$ and source ${}^v \Lambda(x^k, y^4)$. Prescribing two any such functions, we can define (integrating w.r.t. to the coordinate y^4 , or differentiating w.r.t. this coordinate) two other functions of the same type. Such functions can be re-defined in order to transform ${}^v \Lambda(x^k, y^4)$ into an effective constant. The function $\phi(x^i, y^4)$ and/or any its functional can be considered as a generating function which can be prescribed following certain geometric or physical arguments on symmetries, boundary conditions, any explicit singular or non-singular behaviour, smooth class conditions, stochastic conditions, topological configurations etc. This allows us to compute $h_a(x^i, y^4)$ in explicit form. If necessary, we can consider, for instance, the coefficient $h_3(x^i, y^4)$ (or $h_4(x^i, y^4)$) to be a generating function and compute h_4 (or h_3) and ϕ for a given source ${}^v \Lambda(x^k, y^4)$. We note that if we consider vacuum solutions for ${}^s \hat{\mathbf{D}}$ with ${}^v \Lambda = 0$ in (47), we are constrained to study only configurations with N -adapted coefficients $\partial_4 h_3 = 0$ and/or $\partial_4 \phi = 0$. Such solutions are also important to study geometric and physical properties of vacuum off-diagonal configurations and their possible diagonal limits. The decoupling property is explicit for such vacuum and non-vacuum equations because they contain only two coefficients of the metric, h_a ; and the coupling with other diagonal and/or off-diagonal coefficients (like g_i , w_i , n_i , and/or jet prolongations) are not involved.

3. Having computed the coefficient γ (41), the N-connection coefficients n_i can be defined after two integrations on y^4 in (48). This defines a part of N-connection coefficients. A value n_i does not depend on other coefficients of a d-metric except for the coefficient γ determined by h_a .
4. Using h_3 and ϕ from the previous discussion, we compute the coefficients α_i and β , see (41), which allows us to define w_i from the algebraic equations (49). Such w_i define a different part of the N-connection coefficients, can determine off-diagonal coefficients of the metric and may also contribute to the diagonal coefficients if such metrics are written in coordinate bases. Nevertheless, the functions w_i are independent from other coefficients of the d-metric and N-connection with respect to N-adapted frames.
5. The procedure in items 2-4 can be repeated step by step for any shell with r -jet variables. The equations (50)–(52) are completely similar to (47)–(49) but contain additional dependencies on jet coordinates and derivatives ∂ that depend on respective jet coordinates. For instance, the equation (50) and formula (44) with partial derivative ∂_6 are for functions $h_5(x^i, y^a, \zeta^6)$, $h_6(x^i, y^a, \zeta^6)$ and ${}^1\phi(x^i, y^a, \zeta^6)$ and source ${}^v\Lambda(u^\beta, \zeta^6)$. We can compute any two such functions integrating w.r.t. the coordinate ζ^6 if two other ones are prescribed. In a similar form, we follow points 3 and 4 with ${}^1\alpha_\tau$, ${}^1\beta$, ${}^1\gamma$, see (43), and compute the higher order N-connection coefficients ${}^1n_\tau$ and ${}^1w_\tau$.
6. The splitting property holds on any 2-d shell as it is stated in (53). Here we note that it is not clear if any splitting of equations could be proven in general form for 3-d shells. This is partly because the topological properties of 2-d and 3-d shells are very different. The equations of type (47), (50) are degenerate for 1-d shells. That is partly why our AFDM is based on 2+2+2+... splitting which allows to decouple and solve such nonlinear systems of PDEs in general form. We can consider in similar form splitting of type 3+2+2+... for 3-d bases when the point 1 refers to the 3-d Laplace, or d’Alameber equations. For certain configurations, we can generate extra-dimension and jet configurations by imbedding 1,2 and 3 dimensional metrics into some d-metric configurations with the splitting 2+2+2+... In arbitrary systems of reference, such effective vacuum and non-vacuum nonholonomic dynamical systems depend on spacetime coordinates. They may be of nonlinear evolution type, or Ricci soliton fixed configurations, for respectively prescribed signatures. Nevertheless, the assumption on 2-d shall is a condition imposing certain (2-d) topological restrictions on the jet variables extensions.

Finally, we emphasize that the splitting property of nonholonomic and holonomic Einstein equations for higher dimensions was proven in [43]. Geometric and physical models with jet variables and extra dimension coordinates are similar in certain sense but with different jet symmetry conditions.⁹ Formal 2+2+2+... splitting are possible only in adapted nonholonomic systems of reference and the derived nonholonomic dynamical/evolution equations encode a different type of interior gauge like dynamics.

3.3 Jet integral varieties for off-diagonal metrics and generalized connections

The system of nonlinear PDEs (46)–(53) in spacetime and jet variables can be integrated in general forms for any 2-d shell $\dim {}^sV \geq 4$. We note that the coefficients $g_i = \epsilon_i e^{\psi(x^k)}$ are defined by solutions of corresponding Laplace/ d’Alambert equation (46) that do not contain jet coordinates in N-adapted frames. General solutions will be considered for "vertical" spacetime and jet variables.

3.3.1 4-d non-vacuum spacetime nonholonomic configurations

We can solve (47) and (42) for any $\partial_4\phi \neq 0, h_a \neq 0$ and ${}^v\Lambda \neq 0$. Let us re-write respectively the relevant equations,

$$h_3 h_4 = (\partial_4\phi)(\partial_4 h_3)/2 {}^v\Lambda \text{ and } |h_3 h_4| = (\partial_4 h_3)^2 e^{-2\phi}. \quad (54)$$

⁹In this work, the concept of jet symmetry of some classes of solutions for (modified) Einstein equations is used in a sense that it generalizes certain Lie group, Killing type, or anholonomy relations by introducing jet variables. In particular, we obtain elements of the type $L_{n,m}^r$ introduced in section 2.1, and studied in details in Ref. [23, 26, 27].

By considering a new generating function $\Phi := e^\phi$ and introducing the first equation into the second one, we get

$$|\partial_4 h_3| = \frac{\partial_4(e^{2\phi})}{4|{}^v\Lambda|} = \frac{\partial_4[\Phi^2]}{4|{}^v\Lambda|}. \quad (55)$$

Integrating w.r.t. the coordinate y^4 , we find

$$h_3[\Phi, {}^v\Lambda] = {}^0h_3(x^k) + \frac{\epsilon_3\epsilon_4}{4} \int dy^4 \frac{\partial_4(\Phi^2)}{{}^v\Lambda}, \quad (56)$$

where ${}^0h_3 = {}^0h_3(x^k)$ is an integration function and $\epsilon_3, \epsilon_4 = \pm 1$. To compute h_4 we can use the first equation in (54) when

$$h_4[\Phi, {}^v\Lambda] = \frac{(\partial_4\phi)}{{}^v\Lambda} \partial_4(\ln \sqrt{|h_3|}) = \frac{1}{2} \frac{\partial_4\Phi}{{}^v\Lambda} \frac{\partial_4 h_3}{\Phi}. \quad (57)$$

The formulae (56) and (57) for $h_a[\Phi, {}^v\Lambda]$ can be re-parameterized in a more convenient form with an effective cosmological constant $\tilde{\Lambda}_0 = \text{const} \neq 0$. Let us re-define the generating function $\Phi \rightarrow \tilde{\Phi}$, when $\frac{\partial_4[\Phi^2]}{{}^v\Lambda} = \frac{\partial_4[\tilde{\Phi}^2]}{\tilde{\Lambda}_0}$, i.e.

$$\Phi^2 = \tilde{\Lambda}_0^{-1} \int dy^4 ({}^v\Lambda) \partial_4(\tilde{\Phi}^2) \text{ and } \tilde{\Phi}^2 = \tilde{\Lambda}_0 \int dy^4 ({}^v\Lambda)^{-1} \partial_4(\Phi^2). \quad (58)$$

By introducing the integration function ${}^0h_3(x^k)$ and ϵ_3, ϵ_4 , in Φ and respectively, in ${}^v\Lambda$, we express the solutions for h_a as functionals on $[\tilde{\Phi}, \tilde{\Lambda}_0, \Xi]$,

$$h_3[\tilde{\Phi}, \tilde{\Lambda}_0] = \frac{\tilde{\Phi}^2}{4\tilde{\Lambda}_0} \text{ and } h_4[\tilde{\Phi}, \tilde{\Lambda}_0, \Xi] = \frac{(\partial_4\tilde{\Phi})^2}{\Xi}. \quad (59)$$

The functional $\Xi[{}^v\Lambda, \tilde{\Phi}] = \int dy^4 ({}^v\Lambda) \partial_4(\tilde{\Phi}^2)$ in the last formula can be considered as a re-defined source for a prescribed generating function $\tilde{\Phi}, {}^v\Lambda \rightarrow \Xi$, when ${}^v\Lambda = \partial_4\Xi/\partial_4(\tilde{\Phi}^2)$ (it contains information on the Ricci soliton contribution, and/or effective energy-momentum tensor of matter in modified gravity theories). We can work with a couple of generating data, $(\Phi, {}^v\Lambda)$ and $(\tilde{\Phi}, \Xi)$, related by formulae (58) for a prescribed effective cosmological constant $\tilde{\Lambda}_0$.

Using the values h_a (59), we compute the coefficients α_i, β and γ from (41). The resulting solutions for N-coefficients, i.e of respective equations (48) and (49), can be expressed as,

$$\begin{aligned} n_k &= {}_1n_k + {}_2n_k \int dy^4 h_4/(\sqrt{|h_3|})^3 = {}_1n_k + {}_2\tilde{n}_k \int dy^4 (\partial_4\tilde{\Phi})^2/\tilde{\Phi}^3\Xi, \text{ and} \\ w_i &= \partial_i\phi/\partial_4\phi = \partial_i\Phi/\partial_4\Phi = \partial_i\Phi^2/\partial_4\Phi^2 = \int dy^4 \partial_i[({}^v\Lambda)\partial_4(\tilde{\Phi}^2)]/[({}^v\Lambda)\partial_4(\tilde{\Phi}^2)] = \partial_i\Xi/\partial_4\Xi, \end{aligned} \quad (60)$$

where ${}_1n_k(x^i)$ and ${}_2n_k(x^i)$, or ${}_2\tilde{n}_k(x^i) = 8 {}_2n_k(x^i)|\tilde{\Lambda}|^{3/2}$, are integration functions. Putting together the formulae for the coefficients (59)-(60), we prove:

Theorem 3.2 *The system of nonlinear PDEs (46)–(49) for non-vacuum 4-d configurations with Killing symmetry ∂_3 is integrated in general form by quadratic line element of the form $ds_{4[dK]}^2 = g_{\alpha\beta}(x^k, y^4) du^\alpha du^\beta$, when*

$$ds_{4[dK]}^2 = \epsilon_i e^{\psi(x^k)} (dx^i)^2 + \frac{\tilde{\Phi}^2}{4\tilde{\Lambda}_0} [dy^3 + ({}_1n_k + {}_2\tilde{n}_k \int dy^4 \frac{(\partial_4\tilde{\Phi})^2}{\tilde{\Phi}^3\Xi}) dx^k]^2 + \frac{(\partial_4\tilde{\Phi})^2}{\Xi} [dy^4 + \frac{\partial_i\Xi}{\partial_4\Xi} dx^i]^2. \quad (61)$$

This line element defines a family of generic off-diagonal solutions with Killing symmetry $\partial/\partial y^3$ of the 4-d nonholonomic Einstein equations (28)–(30) with source parametrization of the type (45) and for the canonical d-connection $\hat{\mathbf{D}}$ (the label 4dK is for "nonholonomic 4-d Killing solutions"). We can verify by straightforward computations that the nonholonomy coefficients $W_{\alpha\beta}^\gamma$ in (11) are not zero if arbitrary generating function ϕ and integration functions (${}^0h_a, {}_1n_k$ and ${}_2n_k$) are considered. This means that such metrics can not be diagonalized by coordinate transforms in a finite spacetime region. The class of solutions (61) carry nontrivial canonical d-torsion (15) which can be proven by using explicit N-adapted coefficient formulae (A.4) for the canonical d-connection (A.3). In section 3.3.3, we shall state additional conditions when such solutions define LC-configurations. Vacuum nonholonomic spacetime quadratic elements are considered in section A.4.1.

3.3.2 Nonholonomic r -jet prolongations of non-vacuum solutions

The solutions with jet variables can be constructed in certain forms which are similar to the 4-d case but achieved by using new classes of generating and integration functions with dependencies on r -jet shell coordinates. We can generate solutions of the system (50)–(52) with coefficients (44) and (43) following a formal analogy when the generating functions and (effective) sources from the previous paragraph are generalized in the form: $\partial_4 \rightarrow \tilde{\partial}_6, \phi(x^k, y^4) \rightarrow {}^1\phi(u^\tau, \zeta^6), {}^v\Lambda(x^k, y^4) \rightarrow {}^v_1\Lambda(u^\tau, \zeta^6) \dots$ and with re-defined values $\tilde{\Phi}(x^k, y^4) \rightarrow {}^1\tilde{\Phi}(u^\tau, \zeta^6)$ and $\tilde{\Lambda}_0 \rightarrow {}^1\tilde{\Lambda}_0 = \text{const.}$

The first set of r -jet coefficients of the d-metric are computed to be $h_5[{}^1\tilde{\Phi}, {}^1\tilde{\Lambda}] = \frac{{}^1\tilde{\Phi}^2}{4 {}^1\tilde{\Lambda}}$ and $h_6[{}^1\tilde{\Phi}] = \frac{(\tilde{\partial}_6 {}^1\tilde{\Phi})^2}{1\tilde{\Xi}}$, for $1\tilde{\Xi} = \int d\zeta^6 ({}^v_1\Lambda)\tilde{\partial}_6({}^1\tilde{\Phi}^2)$ and, for N-coefficients,

$$\begin{aligned} {}^1n_\tau &= {}^1n_\tau + \frac{1}{2}n_\tau \int d\zeta^6 h_6/(\sqrt{|h_5|})^3 = {}^1n_k + \frac{1}{2}\tilde{n}_k \int d\zeta^6 (\tilde{\partial}_6 {}^1\tilde{\Phi})^2 / ({}^1\tilde{\Phi})^3 {}^1\tilde{\Xi}, \\ {}^1w_\tau &= \partial_\tau {}^1\phi / \tilde{\partial}_6 {}^1\phi = \partial_\tau {}^1\Phi / \tilde{\partial}_6 {}^1\Phi = \partial_\tau {}^1\Xi / \tilde{\partial}_6 {}^1\Xi, \end{aligned}$$

where ${}^0h_{a_1} = {}^0h_{a_1}(u^\tau)$, ${}^1n_k(u^\tau)$ and ${}^1n_k(u^\tau)$ are integration functions.

A general class of quadratic line elements with one shell jet variables defining generic off-diagonal solutions of the nonholonomic canonical deformations of the Einstein equations can be parameterized in the form

$$ds_{4+2[dK]}^2 = ds_{4[dK]}^2 + \frac{{}^1\tilde{\Phi}^2}{4 {}^1\tilde{\Lambda}} \left[d\zeta^5 + \left({}^1n_k + \frac{1}{2}\tilde{n}_k \int d\zeta^6 \frac{(\tilde{\partial}_6 {}^1\tilde{\Phi})^2}{({}^1\tilde{\Phi})^3 {}^1\tilde{\Xi}} \right) du^\tau \right]^2 + \frac{(\tilde{\partial}_6 {}^1\tilde{\Phi})^2}{1\tilde{\Xi}} \left[d\zeta^6 + \frac{\partial_\tau {}^1\Xi}{\tilde{\partial}_6 {}^1\Xi} du^\tau \right]^2, \quad (62)$$

where $ds_{4[dK]}^2$ is given by the formulae (61) and $\tau = 1, 2, 3, 4$. This quadratic line element carries the Killing jet symmetry $\tilde{\partial}_5$ (in N-adapted frames, the metric does not depend on ζ^5).

Extending the constructions to the jet shell $s = 2$ with $\tilde{\partial}_6 \rightarrow \tilde{\partial}_8, {}^1\phi(u^\tau, \zeta^6) \rightarrow {}^2\phi(u^{\tau_1}, \zeta^8), {}^v_1\Lambda(u^\tau, \zeta^6) \rightarrow {}^v_2\Lambda(u^{\tau_1}, \zeta^8) \dots$, with ${}^2\tilde{\Phi}(u^{\tau_1}, \zeta^8), {}^2\tilde{\Lambda}_0$, where $\tau_1 = 1, 2, \dots, 5, 6$, we generate off-diagonal solutions in 8-d jet modified gravity model,

$$ds_{4+4[dK]}^2 = ds_{4+2[dK]}^2 + \frac{{}^2\tilde{\Phi}^2}{4 {}^2\tilde{\Lambda}} \left[d\zeta^7 + \left({}^2n_{\tau_s} + \frac{1}{2}\tilde{n}_k \int d\zeta^8 \frac{(\tilde{\partial}_8 {}^2\tilde{\Phi})^2}{({}^2\tilde{\Phi})^3 {}^2\Xi} \right) du^{\tau_1} \right]^2 + \frac{(\tilde{\partial}_8 {}^2\tilde{\Phi})^2}{2\Xi} \left[d\zeta^8 + \frac{\partial_{\tau_1} {}^2\Xi}{\tilde{\partial}_8 {}^2\Xi} du^{\tau_1} \right]^2, \quad (63)$$

where $ds_{4+2[dK]}^2$ is given by (62), ${}^2\Xi = \int d\zeta^8 ({}^v_2\Lambda)\tilde{\partial}_8({}^2\tilde{\Phi}^2)$, and the corresponding integration/generating functions ${}^0h_{a_2}(u^{\tau_1}); a_2 = 7, 8; {}^1n_{\tau_1}(u^{\tau_1})$ and ${}^2n_{\tau_1}(u^{\tau_1})$ are integration functions.

Using $2+2+\dots$ symmetries of the off-diagonal parameterizations (62) and (63), we can construct exact solutions for arbitrary finite sets of r -jet shells on ${}^s\mathbf{V}$ for a nonholonomic $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$. The corresponding quadratic line elements are

$$\begin{aligned} ds_{4+2s[dK]}^2 &= ds_{2+2s[dK]}^2 + \frac{{}^s\tilde{\Phi}^2}{4 {}^s\tilde{\Lambda}} \left[d\zeta^{3+2s} + \left({}^s n_{\tau_{s-1}} + \frac{1}{2}\tilde{n}_{\tau_{s-1}} \int d\zeta^{4+2s} \frac{(\tilde{\partial}_{4+2s} {}^s\tilde{\Phi})^2}{({}^s\tilde{\Phi})^3 {}^s\Xi} \right) du^{\tau_{s-1}} \right]^2 \\ &+ \frac{(\tilde{\partial}_{4+2s} {}^s\tilde{\Phi})^2}{s\Xi} \left[d\zeta^{4+2s} + \frac{\partial_{\tau_{s-1}} {}^s\Xi}{\tilde{\partial}_{4+2s} {}^s\Xi} du^{\tau_{s-1}} \right]^2, \end{aligned} \quad (64)$$

where ${}^s\Xi = \int d\zeta^{4+2s} ({}^v_s\Lambda)\tilde{\partial}_{4+2s}({}^s\tilde{\Phi}^2)$, and the corresponding integration/generating functions; ${}^s n_{\tau_{s-1}}(u^{\tau_{s-1}})$ and ${}^s n_{\tau_{s-1}}(u^{\tau_{s-1}})$ are also integration functions.

3.3.3 The Levi-Civita conditions

All solutions constructed in this section and (for vacuum configurations) in Appendix define subclasses of generic off-diagonal metrics (40) for the canonical d-connections ${}^s\hat{\mathbf{D}}$ and nontrivial nonholonomically induced d-torsion coefficients $\hat{\mathbf{T}}_{\alpha_s\beta_s}^{\gamma_s}$ (A.4). It is natural to have such torsion fields in theories with gauge like jet symmetries of gravitational and matter fields. Nevertheless, we can perform r -jet prolongations in such forms that the nonholonomic induced torsion vanishes for a subclass of nonholonomic distributions with necessary types

of parameterizations of the generating and integration functions and sources. In explicit form, we construct LC-configurations by imposing additional constraints, including certain "shell by shell jet variables", on the d-metric and N-connection coefficients. By straightforward computations (see details in Refs. [43], and Appendix A.5, for r -jet variables), we verify that if in N-adapted frames

$$\begin{aligned} \text{for } \partial_4 w_i &= \mathbf{e}_i \ln \sqrt{|h_4|}, \mathbf{e}_i \ln \sqrt{|h_3|} = 0, \partial_i w_j = \partial_j w_i \text{ and } \partial_4 n_i = 0; \\ s = 1 : \quad \partial_6 {}^1 w_\alpha &= {}^1 \mathbf{e}_\alpha \ln \sqrt{|h_6|}, {}^1 \mathbf{e}_\alpha \ln \sqrt{|h_5|} = 0, \partial_\alpha {}^1 w_\beta = \partial_\beta {}^1 w_\alpha \text{ and } \partial_6 {}^1 n_\gamma = 0; \\ s = 2 : \quad \partial_8 {}^2 w_{\alpha_1} &= {}^2 \mathbf{e}_{\alpha_1} \ln \sqrt{|h_8|}, {}^2 \mathbf{e}_{\alpha_1} \ln \sqrt{|h_7|} = 0, \partial_{\alpha_1} {}^2 w_{\beta_1} = \partial_{\beta_1} {}^2 w_{\alpha_1} \text{ and } \partial_8 {}^2 n_{\gamma_1} = 0; \\ &\dots \end{aligned} \quad (65)$$

the torsion coefficients vanish. For n -coefficients, such conditions are satisfied if ${}_2 n_k(x^i) = 0$ and $\partial_i {}_1 n_j(x^k) = \partial_j {}_1 n_i(x^k)$; ${}_2 n_\alpha(u^\beta) = 0$ and $\partial_\gamma {}_1 n_\tau(u^\beta) = \partial_\tau {}_1 n_\gamma(u^\beta)$; ${}_2 n_{\alpha_1}(u^{\beta_1}) = 0$ and $\partial_{\gamma_1} {}_2 n_{\tau_1}(u^{\beta_1}) = \partial_{\tau_1} {}_2 n_{\gamma_1}(u^{\beta_1})$ etc. The explicit form of solutions of constraints on w_k derived from (65) depend on the class of vacuum or non-vacuum metrics and their jet prolongations.

Let us find explicit solutions for the LC-conditions (65) using the spacetime coordinates. Such nonholonomic constraints can not be solved in explicit form for arbitrary data $(\Phi, {}^v \Lambda)$, or $(\tilde{\Phi}, \Xi, \tilde{\Lambda}_0)$, and for all types of nonzero integration functions ${}_1 n_j(x^k)$ and ${}_2 n_k(x^i)$. We are able to write such solutions in explicit form if, using coordinate and frame transforms, we fix ${}_2 n_k(x^i) = 0$ and ${}_1 n_j(x^k) = \partial_j n(x^k)$ for a function $n(x^k)$. We use the property that $\mathbf{e}_i \Phi = (\partial_i - w_i \partial_4) \Phi \equiv 0$ for any Φ if $w_i = \partial_i \Phi / \partial_4 \Phi$, see (60). The equality $\mathbf{e}_i H = (\partial_i - w_i \partial_4) H = \frac{\partial H}{\partial \Phi} (\partial_i - w_i \partial_4) \Phi \equiv 0$ holds for any functional $H[\Phi]$. We can restrict our construction to a subclass of generating data $(\Phi, {}^v \Lambda)$ and $(\tilde{\Phi}, \Xi, \tilde{\Lambda}_0)$ that are related via formulae (58) when $H = \tilde{\Phi}[\Phi]$ is a functional which allows us to generate LC-configurations in explicit form. Using $h_3[\tilde{\Phi}] = \tilde{\Phi}^2 / 4\tilde{\Lambda}$ (59) for $H = \tilde{\Phi} = \ln \sqrt{|h_3|}$, we satisfy the second condition, $\mathbf{e}_i \ln \sqrt{|h_3|} = 0$, in (65).

Next, we solve the first condition in (65) for spacetime coordinates. Taking the derivative ∂_4 of $w_i = \partial_i \Phi / \partial_4 \Phi$ (60), we obtain

$$\partial_4 w_i = \frac{(\partial_4 \partial_i \Phi)(\partial_4 \Phi) - (\partial_i \Phi) \partial_4 \partial_4 \Phi}{(\partial_4 \Phi)^2} = \frac{\partial_4 \partial_i \Phi}{\partial_4 \Phi} - \frac{\partial_i \Phi}{\partial_4 \Phi} \frac{\partial_4 \partial_4 \Phi}{\partial_4 \Phi}. \quad (66)$$

Choosing a generating function $\Phi = \check{\Phi}$ for which

$$\partial_4 \partial_i \check{\Phi} = \partial_i \partial_4 \check{\Phi} \quad (67)$$

and using (66), we compute $\partial_4 w_i = \mathbf{e}_i \ln |\partial_4 \check{\Phi}|$. Taking $h_4[\Phi, {}^v \Lambda]$ (57), we write $\mathbf{e}_i \ln \sqrt{|h_4|} = \mathbf{e}_i [\ln |\partial_4 \check{\Phi}| - \ln \sqrt{|{}^v \Lambda|}]$, (see also the conditions (67) and $\mathbf{e}_i \check{\Phi} = 0$). Using the last two formulae, we obtain $\partial_4 w_i = \mathbf{e}_i \ln \sqrt{|h_4|}$ if $\mathbf{e}_i \ln \sqrt{|{}^v \Lambda|} = 0$. This is possible for ${}^v \Lambda = \text{const}$, or if ${}^v \Lambda$ can be expressed as a functional ${}^v \Lambda(x^i, y^4) = {}^v \Lambda[\check{\Phi}]$. Here, we note that the third condition, $\partial_i w_j = \partial_j w_i$, in (65), can be solved for any $\check{A} = \check{A}(x^k, y^4)$ for which $w_i = \check{w}_i = \partial_i \check{\Phi} / \partial_4 \check{\Phi} = \partial_i \check{A}$.

In shell jet variables, we can extend the above constructions for the "shell" generating functions:

$$\begin{aligned} s = 1 : \quad {}^1 \Phi &= {}^1 \check{\Phi}(u^\tau, \zeta^6), \partial_6 \partial_\tau {}^1 \check{\Phi} = \partial_\tau \partial_6 {}^1 \check{\Phi}; \partial_\alpha {}^1 \check{\Phi} / \partial_6 {}^1 \check{\Phi} = \partial_\alpha {}^1 \check{A}; {}^1 n_\tau = \partial_\tau {}^1 n(u^\beta); \\ s = 2 : \quad {}^2 \Phi &= {}^2 \check{\Phi}(u^{\tau_1}, \zeta^8), \partial_8 \partial_{\tau_1} {}^2 \check{\Phi} = \partial_{\tau_1} \partial_8 {}^2 \check{\Phi}; \partial_{\alpha_1} {}^2 \check{\Phi} / \partial_8 {}^2 \check{\Phi} = \partial_{\alpha_2} {}^2 \check{A}; {}^2 n_{\tau_1} = \partial_{\tau_1} {}^2 n(u^{\beta_1}); \dots \end{aligned} \quad (68)$$

We can re-define the generating functions as functionals of the "inverse hat" values, when

$$\begin{aligned} \check{\Phi}^2 &= (\tilde{\Lambda}_0)^{-1} \int dy^4 ({}^v \Lambda) \partial_4 (\tilde{\Phi}^2) \text{ and } \tilde{\Phi}^2 = \tilde{\Lambda} \int dy^4 ({}^v \Lambda)^{-1} \partial_4 (\check{\Phi}^2); \\ {}^1 \check{\Phi}^2 &= ({}^1 \tilde{\Lambda}_0)^{-1} \int d\zeta^6 ({}^v \Lambda) \partial_6 ({}^1 \tilde{\Phi}^2) \text{ and } {}^1 \tilde{\Phi}^2 = {}^1 \tilde{\Lambda} \int d\zeta^6 ({}^v \Lambda)^{-1} \partial_6 ({}^1 \check{\Phi}^2); \\ {}^2 \check{\Phi}^2 &= ({}^2 \tilde{\Lambda}_0)^{-1} \int d\zeta^8 ({}^v \Lambda) \partial_8 ({}^2 \tilde{\Phi}^2) \text{ and } {}^2 \tilde{\Phi}^2 = {}^2 \tilde{\Lambda} \int d\zeta^8 ({}^v \Lambda)^{-1} \partial_8 ({}^2 \check{\Phi}^2), \end{aligned}$$

and compute the values $\Xi(\tilde{\Phi}[\check{\Phi}])$, ${}^1\Xi({}^1\tilde{\Phi}[{}^1\check{\Phi}])$ and ${}^2\Xi({}^2\tilde{\Phi}[{}^2\check{\Phi}])$ as in (63). This way, we construct quadratic line elements for LC-configurations as

$$\begin{aligned}
ds_{4+2s[dK]}^2 = & \epsilon_i e^{\psi(x^k)} (dx^i)^2 + \frac{(\tilde{\Phi}[\check{\Phi}])^2}{4\tilde{\Lambda}_0} [dy^3 + (\partial_i n) dx^i]^2 + \frac{(\partial_4 \tilde{\Phi}[\check{\Phi}])^2}{\Xi(\tilde{\Phi}[\check{\Phi}])} [dy^4 + (\partial_i \check{A}) dx^i]^2 \\
& + \frac{({}^1\tilde{\Phi}[{}^1\check{\Phi}])^2}{4{}^1\tilde{\Lambda}_0} [d\zeta^5 + (\partial_\tau {}^1n) du^\tau]^2 + \frac{(\check{\partial}_6 {}^1\tilde{\Phi}[{}^1\check{\Phi}])^2}{{}^1\Xi({}^1\tilde{\Phi}[{}^1\check{\Phi}])} [d\zeta^6 + (\partial_\tau {}^1\check{A}) du^\tau]^2 \\
& + \frac{({}^2\tilde{\Phi}[{}^2\check{\Phi}])^2}{4{}^2\tilde{\Lambda}_0} [d\zeta^7 + (\check{\partial}_{\tau_1} {}^2n) du^{\tau_1}]^2 + \frac{(\check{\partial}_8 {}^2\tilde{\Phi}[{}^2\check{\Phi}])^2}{{}^2\Xi({}^2\tilde{\Phi}[{}^2\check{\Phi}])} [d\zeta^8 + (\check{\partial}_{\tau_1} {}^2\check{A}) du^{\tau_1}]^2 + \dots \\
& + \frac{({}^s\tilde{\Phi}[{}^s\check{\Phi}])^2}{4{}^s\tilde{\Lambda}_0} [d\zeta^{3+2s} + (\check{\partial}_{\tau_{s-1}} {}^sn) du^{\tau_{s-1}}]^2 + \frac{(\check{\partial}_{2+2s} {}^s\tilde{\Phi}[{}^s\check{\Phi}])^2}{{}^s\Xi({}^s\tilde{\Phi}[{}^s\check{\Phi}])} [d\zeta^{4+2s} + (\check{\partial}_{\tau_{s-1}} {}^s\check{A}) du^{\tau_{s-1}}]^2.
\end{aligned} \tag{69}$$

The torsions of such non-vacuum exact solutions (69) generated by respective data $({}^s\check{\mathbf{g}}, {}^s\check{\mathbf{N}}, {}^s\check{\mathbf{V}})$ are zero, which is different from the class of exact solutions (64) with nontrivial canonical d-torsions (A.4) completely determined by arbitrary data $({}^s\mathbf{g}, {}^s\mathbf{N}, {}^s\hat{\mathbf{D}})$ with Killing symmetry $\check{\partial}_7$. For an arbitrary shell s , we always have a Killing symmetry $\check{\partial}_{s-1}$.

3.4 Violation of Killing symmetries and jet prolongations

Considering prolongations of 4-d nonholonomic Ricci soliton and Einstein equations on jet variables we can generate new classes of solutions with non-Killing symmetries both on spacetime coordinates and on jet shells. On $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$, there are two general possibilities to generate "non-Killing" configurations mentioned in Refs. [43, 15] that in this work are generalized for nonholonomic jet variables: 1) to perform a formal embedding into, for instance, higher dimension jet prolongation of vacuum spacetimes and/or by 2) "vertical" conformal nonholonomic deformations, in general, with jet variables.

3.4.1 Imbedding into a jet prolongation of a vacuum solution

Let us analyze an example when a subclass of off-diagonal metrics for 6-d space with jet variables via nonholonomic constraints and re-parameterizations transform into 4-d non-Killing vacuum solutions. We consider the geometric case: $\Lambda = {}^v\Lambda = {}^1\Lambda = 0$; $h_3 = \epsilon_3, h_5 = \epsilon_5, n_k = 0$ and ${}^1n_\alpha = 0$ with a 2-d h -metric $\epsilon_i e^{\psi(x^k, \Lambda=0)} (dx^i)^2$. The coefficients of the Ricci d-tensor are zero, see formulae (A.5)-(A.8) and (A.9)-(A.11). For such conditions, we can not use equations (46)-(52) derived for $\partial_4 h_3 \neq 0, \check{\partial}_6 h_5 \neq 0$ etc. because such conditions do not allow, for instance, values $h_3 = \epsilon_3, h_5 = \epsilon_5$, for any nontrivial data $h_4(x^i, y^4), w_k(x^i, y^4); h_6(x^i, y^4, \zeta^6), {}^1w_k(x^i, y^4), {}^1w_4(x^i, y^4, \zeta^6)$. Such functions, depending in general, on spacetime and jet variables, can be considered as generating functions for vacuum quadratic line elements

$$ds_{6 \rightarrow 4}^2 = \epsilon_i e^{\psi(x^k, \Lambda=0)} (dx^i)^2 + \epsilon_3 (dy^3)^2 + h_4 (dy^4 + w_k dx^k)^2 + \epsilon_5 (d\zeta^5)^2 + h_6 (d\zeta^6 + {}^1w_k dx^k + {}^1w_4 dy^4)^2 \tag{70}$$

on the first 2-d jet shell on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$. This class of vacuum 6-d metrics with two jet variables are with nonzero nonholonomically induced d-torsion (A.4). Such solutions can not be considered as a subclass of vacuum solutions (A.18) when $h_3 \rightarrow \epsilon_3$ and $h_5 \rightarrow \epsilon_5$ because the conditions $\partial_4 h_3 \neq 0$ and $\check{\partial}_6 h_5 \neq 0$ impose additional constraints on the class of possible generating functions h_4 and h_6 . By fixing from the very beginning certain configurations with $\partial_4 h_3 = 0$ and $\check{\partial}_6 h_5 = 0$, we can consider the values h_4, h_6 and $w_k, {}^1w_k, {}^1w_4$ as independent generating functions.

We generate LC-configurations if the coefficients of the d-metric (70) are subject to additional constraints (65) up to $s = 1$. We can follow a formal procedure which is similar to that outlined in section 3.3.3. For any constant $h_3 = \epsilon_3$ and $h_5 = \epsilon_5$, the conditions $\mathbf{e}_i \ln \sqrt{|h_3|} = 0$ and ${}^1\mathbf{e}_\alpha \ln \sqrt{|h_5|} = 0$ are satisfied. The class of generating functions can be restricted to solve the equations

$$\begin{aligned}
\partial_4 w_i(x^i, y^4) &= \mathbf{e}_i \ln \sqrt{|h_4(x^i, y^4)|}, \partial_i w_j = \partial_j w_i, \text{ and} \\
\check{\partial}_6 {}^1w_\alpha(x^i, y^4, \zeta^6) &= {}^1\mathbf{e}_\alpha \ln \sqrt{|h_6(x^i, y^4, \zeta^6)|}, \partial_\alpha {}^1w_\beta = \partial_\beta {}^1w_\alpha,
\end{aligned} \tag{71}$$

Such equations do not depend on the spacetime coordinate y^3 and on jet variable ζ^5 . By prescribing any values of h_4 and h_6 we can find LC-admissible w -coefficients solving the system of first order partial derivative equations in (71). In general, such solutions are defined for certain nonholonomic constraints, i.e. in "non-explicit" form. If the respective d-metric and N-connection coefficients $h_4[\check{\Phi}]$, $h_6[{}^1\check{\Phi}]$ and $w_k[\check{\Phi}]$, ${}^1w_k[{}^1\check{\Phi}]$, ${}^1w_4[{}^1\check{\Phi}]$ are determined by $\check{\Phi}(x^i, y^4)$ and ${}^1\check{\Phi}(x^i, y^4, \zeta^6)$ and satisfy conditions (67), (68) (for such configurations, h_3 and h_5 may be not functionals of type (59)), then we can solve equations (71) in explicit form.

By choosing any generating function $\check{\Phi}$ or ${}^1\check{\Phi}$ and functionals $h_4[\check{\Phi}]$, $h_6[{}^1\check{\Phi}]$ we compute

$$w_i = \check{w}_i = \partial_i \check{\Phi} / \partial_4 \check{\Phi} = \partial_i \check{A} \text{ and } {}^1w_i = {}^1\check{w}_i = \partial_i {}^1\check{\Phi} / \partial_6 {}^1\check{\Phi} = \partial_i {}^1\check{A}, \quad {}^1w_4 = {}^1\check{w}_4 = \partial_4 {}^1\check{\Phi} / \partial_6 {}^1\check{\Phi} = \partial_4 {}^1\check{A}, \quad (72)$$

for some $\check{A}(x^i, y^4)$ and ${}^1\check{A}(x^i, y^4, \zeta^6)$ which are necessary to satisfy the equalities $\partial_i w_j = \partial_j w_i$ and $\partial_\alpha {}^1w_\beta = \partial_\beta {}^1w_\alpha$. Applying the functional derivatives of type (66) and N-coefficients of type (72) when $H[\check{\Phi}] = \ln \sqrt{|h_4|}$ and ${}^1H[{}^1\check{\Phi}] = \ln \sqrt{|h_6|}$, we can satisfy the LC-conditions (71).

The constructions from the last two paragraphs allow to define a subclass of metrics of (70) determined by generic off-diagonal metrics as solutions of 6-d vacuum Einstein equations with two jet variables from the first shell,

$$ds_{6 \rightarrow 4}^2 = \epsilon_i e^{\psi(x^k, \Lambda=0)} (dx^i)^2 + \epsilon_3 (dy^3)^2 + h_4[\check{\Phi}] (dy^4 + \partial_k \check{A} dx^k)^2 + \epsilon_5 (d\zeta^5)^2 + h_6[{}^1\check{\Phi}] (d\zeta^6 + \partial_k {}^1\check{A} dx^k + \partial_4 {}^1\check{A} dy^4)^2. \quad (73)$$

The terms $\epsilon_3 (dy^3)^2$ and $\epsilon_5 (d\zeta^5)^2$ are for trivial extensions from 4-d to 6-d configurations but imbedded in a nontrivial form in a jet extra dimensional vacuum background. Re-defining the coordinate $\zeta^6 \rightarrow y^3$, we generate vacuum solutions in 4-d gravity with metrics (73) depending on all four coordinates x^i, y^3 and y^4 . This way we mimic certain 4-d gravitational interactions on a jet prolongation of a 3-d spacetime manifold. Finally, we note that the nonholonomy coefficients (11) are not zero and that such metrics can not be diagonalized by coordinate or jet coordinates transformations. This class of 4-d vacuum spacetimes do not possess, in general, any Killing symmetries.

3.4.2 "Vertical" nonholonomic conformal and jet deformations

We briefly touch upon another possibility to generate off-diagonal solutions depending on all spacetime coordinates and, in general, with nontrivial sources of type (A.13) [43]. To work with jet type variable the formal re-definition of extra dimension coordinates into nonholonomic shell jet coordinates is necessary. By straightforward but tedious computations, we can prove

Corollary 3.1 *Any metric*

$$\mathbf{g} = g_i(x^k) dx^i \otimes dx^i + \omega^2(u^\alpha) h_a(x^k, y^4) \mathbf{e}^a \otimes \mathbf{e}^a + {}^1\omega^2(u^{\alpha_1}) h_{a_1}(u^\alpha, \zeta^6) \mathbf{e}^{a_1} \otimes \mathbf{e}^{a_1} + \dots + {}^s\omega^2(u^{\alpha_s}) h_{a_s}(u^{\alpha_{s-1}}, \zeta^{4+2s}) \mathbf{e}^{a_s} \otimes \mathbf{e}^{a_s}, \quad (74)$$

with the conformal v -factors subject to the conditions

$$\begin{aligned} \mathbf{e}_k \omega &= \partial_k \omega + n_k \partial_3 \omega + w_k \partial_4 \omega = 0, \\ {}^1\mathbf{e}_\beta {}^1\omega &= \partial_\beta {}^1\omega + {}^1n_\beta \partial_5 {}^1\omega + {}^1w_\beta \partial_6 {}^1\omega = 0, \quad {}^2\mathbf{e}_{\beta_1} {}^2\omega = \partial_{\beta_1} {}^2\omega + {}^2n_{\beta_1} \partial_7 {}^2\omega + {}^2w_{\beta_1} \partial_8 {}^2\omega = 0, \dots \end{aligned} \quad (75)$$

does not change the Ricci d-tensor (A.5)–(A.12).

As a result of this Corollary, any class of solutions considered in this section can be generalized to non-Killing configurations using "vertical" nonholonomic conformal and jet transformations and deformations.

4 Nonholonomic Jet Prolongations of the Kerr Metric and Ricci Solitons

In this section, we study nonholonomic off-diagonal and/or jet deformations of the Kerr black hole solution. The approach develops the results from section 4 of Ref. [15] for jet variables and Ricci soliton configurations when the constructions for massive gravity are re-considered for jet modified gravity theories. A series of new

class of exact solutions when the metrics are nonholonomically deformed into general or ellipsoidal stationary configurations in four dimensional gravity with Ricci soliton correction and/or extra dimensions treated as jet variables. We cite the monographs [21, 25, 29] for the standard methods and bibliography on stationary black holes.

4.1 N-adapted parameterizations of the Kerr vacuum solution

A 4-d ansatz

$$ds_{[0]}^2 = Y^{-1}e^{2h}(d\rho^2 + dz^2) - \rho^2 Y^{-1}dt^2 + Y(d\varphi + Adt)^2$$

parameterized in terms of three functions (h, Y, A) on coordinates (ρ, z) defines the Kerr solution of the vacuum Einstein equations (for rotating black holes) if we choose

$$Y = \frac{1 - (p\hat{x}_1)^2 - (q\hat{x}_2)^2}{(1 + p\hat{x}_1)^2 + (q\hat{x}_2)^2}, \quad A = 2M \frac{q(1 - \hat{x}_2)(1 + p\hat{x}_1)}{p(1 - (p\hat{x}_1) - (q\hat{x}_2))}, \quad e^{2h} = \frac{1 - (p\hat{x}_1)^2 - (q\hat{x}_2)^2}{p^2[(\hat{x}_1)^2 + (\hat{x}_2)^2]}, \quad \rho^2 = M^2(\hat{x}_1^2 - 1)(1 - \hat{x}_2^2), \quad z = M\hat{x}_1\hat{x}_2,$$

where $M = \text{const}$ and $\rho = 0$ states the horizon $\hat{x}_1 = 0$ with the "north / south" segment of the rotation axis, $\hat{x}_2 = +1/-1$. For our purposes, such a metric is written in the form

$$ds_{[0]}^2 = (dx^1)^2 + (dx^2)^2 - \rho^2 Y^{-1}(\mathbf{e}^3)^2 + Y(\mathbf{e}^4)^2, \quad (76)$$

with some coordinates $x^1(\hat{x}_1, \hat{x}_2)$ and $x^2(\hat{x}_1, \hat{x}_2)$, when $(dx^1)^2 + (dx^2)^2 = M^2 e^{2h}(\hat{x}_1^2 - \hat{x}_2^2)Y^{-1} \left(\frac{d\hat{x}_1^2}{\hat{x}_1^2 - 1} + \frac{d\hat{x}_2^2}{1 - \hat{x}_2^2} \right)$ and $y^3 = t + \hat{y}^3(x^1, x^2)$, $y^4 = \varphi + \hat{y}^4(x^1, x^2, t)$. We write $\mathbf{e}^3 = dt + (\partial_i \hat{y}^3)dx^i$, $\mathbf{e}^4 = dy^4 + (\partial_i \hat{y}^4)dx^i$, for some functions \hat{y}^a , $a = 3, 4$, with $\partial_t \hat{y}^4 = -A(x^k)$.

The Boyer–Lindquist coordinates for the Kerr metric are introduced as $(r, \vartheta, \varphi, t)$, where $r = m_0(1 + p\hat{x}_1)$, $\hat{x}_2 = \cos \vartheta$. The parameters p, q are related to the total black hole mass, m_0 and the total angular momentum, am_0 , for the asymptotically flat, stationary and axisymmetric Kerr spacetime. The formulae $m_0 = Mp^{-1}$ and $a = Mqp^{-1}$ when $p^2 + q^2 = 1$ imply $m_0^2 - a^2 = M^2$. In terms of these variables, the metric (76) is written

$$\begin{aligned} ds_{[0]}^2 &= (dx^{1'})^2 + (dx^{2'})^2 + \overline{A}(\mathbf{e}^{3'})^2 + (\overline{C} - \overline{B}^2/\overline{A})(\mathbf{e}^{4'})^2, \\ \mathbf{e}^{3'} &= dt + d\varphi \overline{B}/\overline{A} = dy^{3'} - \partial_{i'}(\hat{y}^{3'} + \varphi \overline{B}/\overline{A})dx^{i'}, \quad \mathbf{e}^{4'} = dy^{4'} = d\varphi. \end{aligned} \quad (77)$$

In these quadratic elements, we consider coordinate functions $x^{1'}(r, \vartheta)$, $x^{2'}(r, \vartheta)$, $y^{3'} = t + \hat{y}^{3'}(r, \vartheta, \varphi) + \varphi \overline{B}/\overline{A}$, $y^{4'} = \varphi$, $\partial_{\varphi} \hat{y}^{3'} = -\overline{B}/\overline{A}$, for which $(dx^{1'})^2 + (dx^{2'})^2 = \Xi(\Delta^{-1}dr^2 + d\vartheta^2)$, when the coefficients are

$$\begin{aligned} \overline{A} &= -\Xi^{-1}(\Delta - a^2 \sin^2 \vartheta), \quad \overline{B} = \Xi^{-1}a \sin^2 \vartheta [\Delta - (r^2 + a^2)], \\ \overline{C} &= \Xi^{-1} \sin^2 \vartheta [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta], \quad \text{and } \Delta = r^2 - 2m_0r + a^2, \quad \Xi = r^2 + a^2 \cos^2 \vartheta. \end{aligned} \quad (78)$$

We consider the prime data

$$\begin{aligned} \dot{g}_1 &= 1, \dot{g}_2 = 1, \dot{h}_3 = -\rho^2 Y^{-1}, \dot{h}_4 = Y, \dot{N}_i^a = \partial_i \hat{y}^a, \\ \text{i.e. } \dot{g}_{1'} &= 1, \dot{g}_{2'} = 1, \dot{h}_{3'} = \overline{A}, \dot{h}_{4'} = \overline{C} - \overline{B}^2/\overline{A}, \\ \dot{N}_{i'}^3 &= \dot{n}_{i'} = -\partial_{i'}(\hat{y}^{3'} + \varphi \overline{B}/\overline{A}), \dot{N}_{i'}^4 = \dot{w}_{i'} = 0 \end{aligned} \quad (79)$$

for the quadratic linear elements (76), or (77), which define exact solutions with rotational spherical symmetry of the vacuum Einstein equations parameterized in the form (25) and (26) with zero sources. The Kerr vacuum solution in 4-d GR consists of a "degenerate" case of 4-d off-diagonal vacuum solutions determined by primary metrics with data (79) when the diagonal coefficients depend only on two "horizontal" N-adapted coordinates and the off-diagonal terms are induced by rotating frames.

4.2 Deformations of Kerr metrics by an effective Ricci soliton source

Let us consider the coefficients (79) for the Kerr metric as the data for a prime metric $\mathring{\mathbf{g}}$. Our goal is to study nonholonomic off-diagonal deformations of the Kerr solution into a Ricci soliton configuration, i.e. when the vacuum Einstein equations are modified by a Ricci soliton, with

$$(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^v\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \rightarrow (\tilde{\mathbf{g}}, \tilde{\mathbf{N}}, {}^v\tilde{\Upsilon} = \Lambda(x^k), {}^h\tilde{\Upsilon} = {}^v\Lambda(x^k, y^4)), \tilde{\Lambda}_0 = \text{const} \neq 0,$$

where the target source (45) is parameterized as $\tilde{\Upsilon}_1^1 = \tilde{\Upsilon}_2^2 = {}^h\tilde{\Upsilon} = {}^v\Lambda(x^k, y^4)$ and $\Upsilon_3^3 = \Upsilon_4^4 = {}^v\Upsilon = \Lambda(x^k)$ and encode contributions of gradient function κ and the constant λ from the Ricci soliton equations (20) into solutions of equations (28)–(30). The target metric $\tilde{\mathbf{g}}$ is constrained to define a generic off-diagonal solution of the field equations with effective horizontal (h)- and vertical (v)-polarized gravitational constants. In some sense, the Ricci soliton contributions may induce a mass term of the type $\tilde{\Lambda}_0 = \mu_g^2 \tilde{\lambda}$, like the one considered in [15], for respective parameterizations. The N-adapted deformations of coefficients of metrics and frames are written as

$$[\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i] \rightarrow [\tilde{g}_i = \tilde{\eta}_i \mathring{g}_i, \tilde{h}_3 = \tilde{\eta}_3 \mathring{h}_3, \tilde{h}_4 = \tilde{\eta}_4 \mathring{h}_4, \tilde{w}_i = \mathring{w}_i + {}^\eta w_i, n_i = \mathring{n}_i + {}^\eta n_i],$$

where the values $\tilde{\eta}_a, \tilde{w}_i, \tilde{n}_i$ and ϖ are functions of three coordinates $(x^{k'}, y^4 = \varphi)$ and $\tilde{\eta}_i(x^k)$ and depend only on h-coordinates x^k . The prime data $\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i$ for a Kerr metric are given by coefficients depending only on (x^k) . The quadratic line elements, determined by target solutions of type (61), are parameterized in the form

$$ds_{4[dK]}^2 = e^{\psi(x^{k'})} [(dx^{1'})^2 + (dx^{2'})^2] - \frac{\tilde{\Phi}^2}{4\tilde{\Lambda}_0} \left[dy^3 + \left({}_1n_k + {}_2\tilde{n}_k \int d\varphi \frac{(\partial_\varphi \tilde{\Phi})^2}{\tilde{\Phi}^3 \Xi} \right) dx^k \right]^2 + \frac{(\partial_4 \tilde{\Phi})^2}{\Xi} \left[d\varphi + \frac{\partial_i \Xi}{\partial_\varphi \Xi} dx^i \right]^2,$$

where $\Xi[{}^v\Lambda, \tilde{\Phi}] = \int d\varphi ({}^v\Lambda) \partial_\varphi (\tilde{\Phi}^2)$.

In terms of η -functions (39) giving $h_a^* \neq 0$, $g_i = c_i e^{\psi(x^{k'})}$ and LC-configurations, the solutions of type (61) with an effective cosmological constant Λ_0 induced by off-diagonal Ricci soliton configurations and ${}_2n_{k'} = 0$ can be re-written in the form

$$ds^2 = e^{\psi(x^{k'})} [(dx^{1'})^2 + (dx^{2'})^2] - \tilde{\eta}_{3'} \overline{A} [dy^{3'} + \left(\partial_{k'} {}^\eta n(x^{i'}) - \partial_{k'} (\hat{y}^{3'} + \varphi \overline{B}/\overline{A}) \right) dx^{k'}]^2 + \tilde{\eta}_{4'} (\overline{C} - \overline{B}^2/\overline{A}) [d\varphi + (\partial_{i'} {}^\eta \tilde{A}) dx^{i'}]^2, \quad (80)$$

where use is made of "primed" coordinates and prime Kerr data (77) and (79). The gravitational polarizations (η_i, η_a) and N-coefficients (n_i, w_i) are computed

$$\begin{aligned} e^{\psi(x^k)} &= \tilde{\eta}_{1'} = \tilde{\eta}_{2'}, \tilde{\eta}_{3'} = \tilde{\Phi}^2 / 4\tilde{\Lambda}_0 \overline{A}, \tilde{\eta}_{4'} = (\partial_\varphi \tilde{\Phi})^2 / \Xi (\overline{C} - \overline{B}^2/\overline{A}), \\ w_{i'} &= \mathring{w}_{i'} + {}^\eta w_{i'} = \partial_{i'} ({}^\eta \tilde{A}[\tilde{\Phi}]), n_{k'} = \mathring{n}_{k'} + {}^\eta n_{k'} = \partial_{k'} (-\hat{y}^{3'} + \varphi \overline{B}/\overline{A} + {}^\eta n), \end{aligned} \quad (81)$$

where ${}^\eta \tilde{A}(x^k, \varphi)$ is introduced via formulae and assumptions similar to (68), for $s = 1$, and $\psi(x^k)$ is a solution of 2-d Poisson equation,

$$\partial_{11}^2 \psi + \partial_{22}^2 \psi = 2 \Lambda(x^{k'}).$$

To extract LC-configurations, the parameterizations (60) are made use of when $\mathring{h}_{3'} \mathring{h}_{4'} = \overline{AC} - \overline{B}^2$ and the N-coefficients are computed as

$$w_{i'} = \mathring{w}_{i'} + {}^\eta w_{i'} = \partial_{i'} (\tilde{\Phi} \sqrt{|\overline{AC} - \overline{B}^2|}) / \partial_\varphi \tilde{\Phi} \sqrt{|\overline{AC} - \overline{B}^2|} = \partial_{i'} {}^\eta \tilde{A}$$

for ${}_1n_{i'} = \partial_{i'} {}^\eta n(x^k)$ computed for an arbitrary function ${}^\eta n(x^k)$.

Theorem 4.1 *Quadratic elements (80) define nonholonomic deformations of a prime Kerr solution $[\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i]$ (79) into target Ricci soliton LC-configurations with Killing symmetry $\partial/\partial \hat{y}^{3'}$ determined by polarization functions (81) generated by data $[\psi(x^{k'}), \tilde{\eta}_{3'}(x^{k'}, \varphi), {}^\eta \tilde{A}(\tilde{\eta}_{3'}), {}^\eta n(x^k), {}^v\Lambda(x^{k'}, \varphi), \tilde{\Lambda}_0]$.*

Proof. Let us show that $\tilde{\eta}_{4'}$ can be defined by $\tilde{\eta}_{3'}$ which can be considered as a generating function instead of $\tilde{\Phi}$. Considering the second formula in (81), we express

$$\tilde{\Phi}^2 = 4\tilde{\Lambda}_0 \bar{A} \tilde{\eta}_{3'}$$

and compute $\Xi = 4\tilde{\Lambda}_0 \bar{A} \int d\varphi ({}^v\Lambda) \partial_\varphi(\tilde{\eta}_{3'})$. We introduce these formulae into the third formula in (81) and derive

$$\tilde{\eta}_{4'} = \bar{A} \left(\partial_\varphi \sqrt{|\tilde{\eta}_{3'}|} \right)^2 / (\bar{A}C - \bar{B}^2) \int d\varphi ({}^v\Lambda) \partial_\varphi(\tilde{\eta}_{3'}).$$

It follows that, by prescribing any polarization function $\tilde{\eta}_{3'}(x^{k'}, \varphi)$ and v -source ${}^v\Lambda(x^{k'}, \varphi)$, we can compute $\tilde{\eta}_{4'}$. The polarizations $\tilde{\eta}_{1'} = \tilde{\eta}_{2'}$ are determined by function $\psi(x^{k'})$, i.e. by source $\Lambda(x^{k'})$. Finally, by prescribing any functional ${}^\eta\tilde{A}(\tilde{\eta}_{3'})$ and function ${}^\eta n(x^k)$ we can compute the N-connection coefficients for any fixed effective cosmological constant $\tilde{\Lambda}_0$.

□

The solutions (80) are for stationary LC-configurations, generated canonically as off-diagonal Ricci solitons from Kerr black holes when the new class of spacetimes carry Killing symmetry $\partial/\partial y^{3'}$ and generic dependence on three (from maximally four) coordinates, $(x^{i'}(r, \vartheta), \varphi)$. Off-diagonal modifications are possible even for very small values of the effective cosmological constant which can mimic gravitational effects determined by a gravitational mass parameter μ_g .

4.2.1 Nonholonomically induced torsion and Ricci soliton modified gravity

If we do not impose the LC-conditions (26), a nontrivial source ${}^v\Lambda(x^{k'}, \varphi)$ induces stationary configuration with nontrivial d-torsion (A.4). For simplicity, we can study nonholonomic torsion effects for a v -source not depending on the coordinate φ , i.e. for ${}^v\Lambda(x^{k'})$. The torsion coefficients are determined by metrics of the type (61) with nontrivial $\tilde{\Lambda}_0$ and certain parameterizations of coefficients of an associated N-connection, canonical d-torsion and coordinates distinguishing the prime data for a Kerr metric (79). The corresponding quadratic elements can be written in the form

$$\begin{aligned} ds^2 = & e^{\psi(x^{k'})} [(dx^{1'})^2 + (dx^{2'})^2] - \frac{\Phi^2}{4|\tilde{\Lambda}_0|} \bar{A} [dy^{3'} + \left({}_1n_{k'}(x^{i'}) + {}_2n_{k'}(x^{i'}) \frac{(\partial_\varphi \Phi)^2}{\Phi^5} - \partial_{k'}(\hat{y}^{3'} + \varphi \bar{B}/\bar{A}) \right) dx^{k'}]^2 \\ & + \frac{(\partial_\varphi \Phi)^2}{{}^v\Lambda(x^{k'}) \Phi^2} (\bar{C} - \bar{B}^2/\bar{A}) [d\varphi + \frac{\partial_{i'} \Phi}{\partial_\varphi \Phi} dx^{i'}]^2, \end{aligned} \quad (82)$$

where nonzero values of ${}_2n_k(x^{i'})$ are considered. We can see that Ricci soliton effects may give nontrivial stationary off-diagonal torsion effects if the integration function ${}_2n_k \neq 0$. Considering two different classes of off-diagonal solutions (82) and (80), we can study the issue if a Ricci modified gravity theory carries induced torsion or is characterized by additional nonholonomic constraints as in GR (giving zero torsion).

It should be noted that configurations of type (82) can be constructed in various theories with noncommutative, brane, extra-dimension, warped and trapped brane type variables in string, or Finsler like and/or Hořava–Lifshits theories [40, 43, 45, 15] when nonholonomically induced torsion effects are significant/non-vanishing.

4.2.2 Small Ricci soliton modifications of Kerr metrics and modelling modified and massive gravity

We can construct off-diagonal solutions for superposition of Ricci soliton effects and f -modified and massive gravity interactions, see original contributions and reviews of results in Refs. [8, 31, 32, 14, 35, 19, 20, 24, 5, 28, 15]. Small nonlinear effects and modifications can be distinguished in explicit form if we take into account additional f -deformations, for instance, a "prime" solution for massive gravity/ effective modeled in GR with source ${}^\mu\Lambda = \mu_g^2 \lambda(x^{k'})$, or re-defined to ${}^\mu\tilde{\Lambda} = \mu_g^2 \tilde{\lambda} = \text{const.}$ By adding a "small" value $\tilde{\Lambda}$ as determined by f -modifications, we work in N-adapted frames with an effective source $\Upsilon = \tilde{\Lambda} + \tilde{\lambda}$. We construct a class

of off-diagonal solutions in modified f -gravity generated from the Kerr black hole solution as a result of two nonholonomic deformations

$$(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^v\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \rightarrow (\tilde{\mathbf{g}}, \tilde{\mathbf{N}}, {}^v\tilde{\Upsilon} = \tilde{\lambda}, \tilde{\Upsilon} = \tilde{\lambda}) \rightarrow ({}^\varepsilon\mathbf{g}, {}^\varepsilon\mathbf{N}, \Upsilon = {}^\varepsilon\tilde{\Lambda} + {}^\mu\tilde{\Lambda}, {}^v\Upsilon = {}^\varepsilon\tilde{\Lambda} + {}^\mu\tilde{\Lambda}),$$

when the target data $\mathbf{g} = {}^\varepsilon\mathbf{g}$ and $\mathbf{N} = {}^\varepsilon\mathbf{N}$ depend on a small parameter ε , $0 < \varepsilon \ll 1$. For simplicity, we construct generic off-diagonal solutions with $|\varepsilon\tilde{\Lambda}| \ll |{}^\mu\tilde{\Lambda}|$, when f -modifications in N -adapted frames are much smaller than massive gravity effects. A similar analysis for nonlinear interactions with $|\varepsilon\tilde{\Lambda}| \gg |{}^\mu\tilde{\Lambda}|$ is omitted. The corresponding N -adapted transforms are parameterized as

$$\begin{aligned} &[g_i, h_a, w_i, n_i] \rightarrow \\ &[g_i = (1 + \varepsilon\chi_i)\tilde{\eta}_i\mathring{g}_i, h_3 = (1 + \varepsilon\chi_3)\tilde{\eta}_3\mathring{h}_3, h_4 = (1 + \varepsilon\chi_4)\tilde{\eta}_4\mathring{h}_4, {}^\varepsilon w_i = \mathring{w}_i + \tilde{w}_i + \varepsilon\bar{w}_i, {}^\varepsilon n_i = \mathring{n}_i + \tilde{n}_i + \varepsilon\bar{n}_i]; \\ &\Upsilon = {}^\mu\tilde{\Lambda}(1 + \varepsilon\tilde{\Lambda}/{}^\mu\tilde{\Lambda}); \quad {}^\varepsilon\tilde{\Phi} = \tilde{\Phi}(x^k, \varphi)[1 + \varepsilon {}^1\tilde{\Phi}(x^k, \varphi)/\tilde{\Phi}(x^k, \varphi)] = \exp[{}^\varepsilon\varpi(x^k, \varphi)], \end{aligned} \quad (83)$$

$$ds_{4\text{edK}}^2 = \varepsilon_i(1 + \varepsilon\chi_i)e^{\psi(x^k)}(dx^i)^2 + \frac{{}^\varepsilon\tilde{\Phi}^2}{4\Upsilon} [dy^3 + (\partial_i n)dx^i]^2 + \frac{(\partial_\varphi {}^\varepsilon\tilde{\Phi})^2}{\Upsilon {}^\varepsilon\tilde{\Phi}^2} [dy^4 + (\partial_i {}^\varepsilon\tilde{A})dx^i]^2,$$

which for LC-configurations, $\partial_i {}^\varepsilon\tilde{A} = \partial_i {}^\varepsilon\tilde{A} + \varepsilon\partial_i {}^1\tilde{A}$ is determined by ${}^\varepsilon\tilde{\Phi} = \tilde{\Phi} + \varepsilon {}^1\tilde{\Phi}$ following conditions (72). The values labeled by "o" and "n" are taken as in previous sections but, for simplicity, we omit priming of indices and consider $\varepsilon\bar{n}_i = 0$. The χ - and w -values are computed for ε -deformed LC-configurations, see formulae (65) for spacetime components, as solutions of the system (A.13) in the form (46)–(49) for a source $\Upsilon = {}^\mu\tilde{\Lambda} + \varepsilon\tilde{\Lambda}$.

The nonholonomic deformations (83) of the off-diagonal metrics (80) give a new class of ε -deformed solutions with

$$\begin{aligned} \chi_1 &= \chi_2 = \chi, \text{ for } \partial_{11}^2\chi + \varepsilon_2\partial_{22}^2\chi = 2\tilde{\Lambda}; \\ \chi_3 &= 2 {}^1\tilde{\Phi}/\tilde{\Phi} - \tilde{\Lambda}/{}^\mu\tilde{\Lambda}, \chi_4 = 2\partial_4 {}^1\tilde{\Phi}/\tilde{\Phi} - 2 {}^1\tilde{\Phi}/\tilde{\Phi} - \tilde{\Lambda}/{}^\mu\tilde{\Lambda}, \bar{w}_i = \left(\frac{\partial_i {}^1\tilde{\Phi}}{\partial_i \tilde{\Phi}} - \frac{\partial_4 {}^1\tilde{\Phi}}{\partial_4 \tilde{\Phi}}\right)\frac{\partial_i \tilde{\Phi}}{\partial_4 \tilde{\Phi}} = \partial_i {}^1\tilde{A}, \bar{n}_i = 0. \end{aligned} \quad (84)$$

There is no summation on index "i" in the last formula and $\mathring{h}_3\mathring{h}_4 = \overline{AC} - \overline{B}^2$. The deformations are determined respectively by two generating functions $\tilde{\Phi}$ and ${}^1\tilde{\Phi}$ and two sources ${}^\mu\tilde{\Lambda}$ and $\tilde{\Lambda}$.

Summarizing the results, we construct an off-diagonal generalization of the Kerr metric by Ricci solitons, "main" mass gravity terms and additional ε -parametric f -modifications,

$$\begin{aligned} ds^2 &= e^{\psi(x^{k'})}(1 + \varepsilon\chi(x^{k'}))[(dx^{1'})^2 + (dx^{2'})^2] - \\ &\frac{\tilde{\Phi}^2}{4|{}^\mu\tilde{\Lambda}|}\overline{A}[1 + \varepsilon(2 {}^1\tilde{\Phi}/\tilde{\Phi} - \tilde{\Lambda}/{}^\mu\tilde{\Lambda})][dy^{3'} + \left(\partial_{k'} n(x^{i'}) - \partial_{k'}(\hat{y}^{3'} + \varphi\overline{B}/\overline{A})\right)dx^{k'}]^2 + \\ &\frac{(\partial_\varphi \tilde{\Phi})^2}{{}^\mu\tilde{\Lambda}\tilde{\Phi}^2}(\overline{C} - \overline{B}^2/\overline{A})[1 + \varepsilon(2\partial_4 {}^1\tilde{\Phi}/\tilde{\Phi} - 2 {}^1\tilde{\Phi}/\tilde{\Phi} - \tilde{\Lambda}/{}^\mu\tilde{\Lambda})][d\varphi + (\partial_{i'} \tilde{A} + \varepsilon\partial_{i'} {}^1\tilde{A})dx^{i'}]^2. \end{aligned} \quad (85)$$

We can consider ε -deformations of type (83) for (82) and generate new classes of off-diagonal solutions with nonholonomically induced torsion determined both by Ricci soliton, massive and f -modifications of GR. Such geometric and physical models are new and can not be identified with effective ones with anisotropic polarizations in GR which also give different r -jet symmetries and prolongations.

4.3 Nonholonomic r -jet off-diagonal Ricci soliton prolongations of the Kerr solution

In reference [15], we studied generic off-diagonal deformations of the Kerr metric into solutions on higher dimensional spacetimes. The goal of this section is to show how prolongations on r -jet variables can be performed following similar methods but generalized to include nonholonomic jet variables.

4.3.1 Jet one shell deformations with nontrivial cosmological constant

Jet symmetries impose certain constraints on possible off-diagonal deformations of a Kerr metric generalized for a corresponding class of solutions with any nontrivial cosmological constant in 6-d. (In a similar form we can generalize the constructions for any finite number of shells). The corresponding class of Kerr – de Sitter jet prolongation configurations are generated by nonholonomic deformations $(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^v\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \rightarrow (\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^v\mathring{\Upsilon} = \Lambda, \mathring{\Upsilon} = \Lambda, {}^{v_1}\mathring{\Upsilon} = \Lambda)$ when solutions are characterized by a jet Killing symmetry $\mathring{\partial}/\partial\zeta^5$ and parameterized as

$$ds^2 = e^{\psi(x^{k'})}[(dx^{1'})^2 + (dx^{2'})^2] - \frac{\tilde{\Phi}^2}{4\Lambda}\overline{A}[dy^{3'} + \left(\partial_{k'} {}^n n(x^{i'}) - \partial_{k'}(\hat{y}^{3'} + \varphi\overline{B}/\overline{A})\right)dx^{k'}]^2 + \frac{(\partial_\varphi\tilde{\Phi})^2}{\Lambda\tilde{\Phi}^2}(\overline{AC} - \overline{B}^2)[d\varphi + (\partial_{i'} {}^n \tilde{A})dx^{i'}]^2 + \frac{{}^1\tilde{\Phi}^2}{4\Lambda}[d\zeta^5 + (\partial_\tau {}^1 n)du^\tau]^2 + \frac{(\mathring{\partial}_6 {}^1\tilde{\Phi})^2}{\Lambda {}^1\tilde{\Phi}^2}[d\zeta^6 + (\partial_\tau {}^1 \tilde{A})du^\tau]^2. \quad (86)$$

The generating functions for such d-metrics are parameterized as

$$\tilde{\Phi} = \tilde{\Phi}(x^{k'}, \varphi), {}^1\tilde{\Phi}(u^\beta, \zeta^6) = {}^1\tilde{\Phi}(x^{k'}, t, \varphi, \zeta^6); {}^n n = {}^n n(x^{i'}), {}^1 n = {}^1 n(u^\beta, \zeta^6); {}^n \tilde{A} = {}^n \tilde{A}(x^{k'}, \varphi), {}^1 \tilde{A} = {}^1 \tilde{A}(u^\beta, \zeta^6),$$

and subject to LC-conditions and conditions of integrability and the "primary" data $\overline{A}, \overline{B}, \overline{C}$ are taken for the Kerr solution in the form (78).

By imposing additional symmetries and constraints on the spacetime generating functions, we can "extract" ellipsoid configurations for a subclass of metrics with ε -deformations,

$$ds^2 = e^{\psi(x^{k'})}[(dx^{1'})^2 + (dx^{2'})^2] - \frac{\tilde{\Phi}^2}{4\Lambda}\overline{A}[1 + 2\varepsilon\zeta\sin(\omega_0\varphi + \varphi_0)][dy^{3'} + \left(\partial_{k'} {}^n n(x^{i'}) - \partial_{k'}(\hat{y}^{3'} + \varphi\frac{\overline{B}}{\overline{A}})\right)dx^{k'}]^2 + \frac{(\partial_\varphi\tilde{\Phi})^2}{\Lambda\tilde{\Phi}^2}(\overline{C} - \overline{B}^2/\overline{A})[1 + \varepsilon(2\frac{\partial_\varphi\tilde{\Phi}}{\tilde{\Phi}}\zeta\sin(\omega_0\varphi + \varphi_0) + 2\omega_0\zeta\cos(\omega_0\varphi + \varphi_0))][d\varphi + (\partial_{i'} {}^n \tilde{A})dx^{i'}]^2 + \frac{{}^1\tilde{\Phi}^2}{4\Lambda}[d\zeta^5 + (\partial_\tau {}^1 n)du^\tau]^2 + \frac{(\mathring{\partial}_6 {}^1\tilde{\Phi})^2}{\Lambda {}^1\tilde{\Phi}^2}[d\zeta^6 + (\partial_\tau {}^1 \tilde{A})du^\tau]^2,$$

where ζ, ω_0 and φ_0 are certain constants determining gravitational rotoid configurations with eccentricity ε . For small values of ε , such metrics describe "slightly" deformed Kerr black holes embedded self-consistently into a generic off-diagonal jet prolongation as a 6-d spacetime.

4.3.2 Two shell effective 8-d jet prolongations

Applying the AFDM, we can construct two shell nonholonomic jet prolongations of the Kerr metric which, in general, are with nontrivial induced torsion for an effective 8-d spacetime with interior jet symmetries. The nonholonomic deformations are defined by the data $(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^v\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \rightarrow (\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^v\mathring{\Upsilon} = \Lambda, \mathring{\Upsilon} = \Lambda, {}^{v_1}\mathring{\Upsilon} = \Lambda, {}^{v_2}\mathring{\Upsilon} = \Lambda)$ and extending on jet variables the 4-d quadratic element (82) but for a different source (we consider a cosmological constant Λ for all dimensions). The corresponding class of solutions is determined by

$$ds^2 = e^{\psi(x^{k'})}[(dx^{1'})^2 + (dx^{2'})^2] - \frac{\Phi^2}{4\Lambda}\overline{A}[dy^{3'} + \left({}_1 n_{k'}(x^{i'}) + {}_2 n_{k'}(x^{i'})\frac{(\partial_\varphi\Phi)^2}{\Phi^5} - \partial_{k'}(\hat{y}^{3'} + \varphi\frac{\overline{B}}{\overline{A}})\right)dx^{k'}]^2 + \frac{(\partial_\varphi\Phi)^2}{\Lambda\Phi^2\overline{A}}(\overline{AC} - \overline{B}^2)[d\varphi + \frac{\partial_{i'}\Phi}{\partial_\varphi\Phi}dx^{i'}]^2 + \frac{{}^1\tilde{\Phi}^2}{4\Lambda}[d\zeta^5 + (\partial_\tau {}^1 n)du^\tau]^2 + \frac{(\mathring{\partial}_6 {}^1\tilde{\Phi})^2}{\Lambda {}^1\tilde{\Phi}^2}[d\zeta^6 + (\partial_\tau {}^1 \tilde{A})du^\tau]^2 + \frac{{}^2\tilde{\Phi}^2}{4\Lambda}[d\zeta^7 + (\partial_{\tau_1} {}^2 n)du^{\tau_1}]^2 + \frac{(\mathring{\partial}_8 {}^2\tilde{\Phi})^2}{\Lambda {}^2\tilde{\Phi}^2}[d\zeta^8 + (\partial_{\tau_1} {}^2 \tilde{A})du^{\tau_1}]^2. \quad (87)$$

The generating functions depend on spacetime and jet variables,

$$\Phi = \Phi(x^{k'}, \varphi), {}^1\tilde{\Phi}(u^\beta, \zeta^6) = {}^1\tilde{\Phi}(x^{k'}, t, \varphi, \zeta^6), {}^2\tilde{\Phi}(u^{\beta_1}, \zeta^8) = {}^2\tilde{\Phi}(x^{k'}, t, \varphi, \zeta^5, \zeta^6, \zeta^8); {}^1 n = {}^1 n(u^\beta, \zeta^6), {}^2 n = {}^2 n(u^{\beta_1}, \zeta^8), {}^n \tilde{A} = {}^n \tilde{A}(x^{k'}, \varphi), {}^1 \tilde{A} = {}^1 \tilde{A}(x^{k'}, t, \varphi, \zeta^6), {}^2 \tilde{A} = {}^2 \tilde{A}(x^{k'}, t, \varphi, \zeta^5, \zeta^6, \zeta^8). \quad (88)$$

Such values are chosen in such forms when the nonholonomically induced torsion (A.4) is effectively modeled on a 4-d pseudo-Riemannian spacetime but on jet shells $s = 1$ and $s = 2$ the torsion fields are zero. This mean

that there are jet coordinate transforms to certain classes of holonomic variables. We can generate jet depending nontrivial torsion N-adapted coefficients if nontrivial integration functions of type ${}_2n_{k'}(x^{i'})$ are extended to contain jet variables.

4.3.3 Kerr Ricci soliton deformations and vacuum r -jet prolongations

Classes of solutions exist with jet variables describing vacuum ellipsoid spacetime configurations with prolongations on two shell jet variables when the source is of type $\Upsilon = \tilde{\Lambda} + \varepsilon(\tilde{\Lambda} + \Lambda) = 0$, with effective massive gravity term ${}^\mu\tilde{\Lambda} = \mu_g^2 |\lambda|$, and give ellipsoidal off-diagonal configurations in GR. For such metrics, $\varepsilon = -{}^\mu\tilde{\Lambda}/(\tilde{\Lambda} + \Lambda) \ll 1$ can be considered as an eccentricity parameter. The corresponding models of off-diagonal jet interior gravitational interactions are with f -modifications when $\tilde{\Lambda}$ compensates nonholonomic contributions via effective constant $\tilde{\Lambda}$ and relates the constructions to massive gravity deformations of a Kerr solution. This subclass of solutions for ε -deformations into vacuum solutions is parameterized by target ansatz

$$\begin{aligned}
ds^2 = & e^{\psi(x^{k'})}(1 + \varepsilon\chi(x^{k'}))[(dx^{1'})^2 + (dx^{2'})^2] - \frac{\tilde{\Phi}^2}{4} \frac{\bar{A}}{\mu\tilde{\Lambda}} [1 + \varepsilon\chi_{3'}][dy^{3'} + (\partial_{k'} {}^\eta n(x^{i'}) - \partial_{k'}(\hat{y}^{3'} + \varphi\bar{B}/\bar{A}))dx^{k'}]^2 + \\
& \frac{(\partial_\varphi\tilde{\Phi})^2\eta_{4'}}{\mu\tilde{\Lambda}\tilde{\Phi}^2}(\bar{C} - \frac{\bar{B}^2}{\bar{A}})[1 + \varepsilon\chi_{4'}][d\varphi + (\partial_{i'}\tilde{A} + \varepsilon\partial_{i'}{}^1\tilde{A})dx^{i'}]^2 + \frac{{}^1\tilde{\Phi}^2}{4(\tilde{\Lambda} + \Lambda)}[d\zeta^5 + (\partial_\tau{}^1n)du^\tau]^2 + \\
& \frac{(\partial_6{}^1\tilde{\Phi})^2}{(\tilde{\Lambda} + \Lambda){}^1\tilde{\Phi}^2}[d\zeta^6 + (\partial_\tau{}^1\tilde{A})du^\tau]^2 + \frac{{}^2\tilde{\Phi}^2}{4(\tilde{\Lambda} + \Lambda)}[d\zeta^7 + (\partial_{\tau_1}{}^2n)du^{\tau_1}]^2 + \frac{(\partial_8{}^2\tilde{\Phi})^2}{(\tilde{\Lambda} + \Lambda){}^2\tilde{\Phi}^2}[d\zeta^8 + (\partial_{\tau_1}{}^2\tilde{A})du^{\tau_1}]^2.
\end{aligned} \tag{89}$$

The jet components are generated by functions ${}^1\tilde{\Phi}$, ${}^2\tilde{\Phi}$ and N-coefficients similar to solutions (87) but with modified effective jet prolongation sources, $\Lambda \rightarrow \tilde{\Lambda} + \Lambda$. This result shows that interior jet interactions can mimic ε -deformations in order to compensate contributions from f -modifications and even the effective vacuum configurations for the 4-d horizontal part. In general, vacuum metrics (89) encode jet modifications/ polarizations of physical constants and coefficients of metrics under nonlinear polarizations of an effective 8-d vacuum distinguishing 4-d nonholonomic configurations and Ricci soliton or massive gravity contributions. Jet variables and f -modified contributions are described by terms proportional to eccentricity parameter ε .

4.3.4 Jet ellipsoid like Kerr – de Sitter configurations

Using the solutions (87), we can construct a class of non-vacuum 8-d jet prolonged solutions with rotoid configurations. For this, we choose for ε -deformations (see a similar formula (84) for 4-d) a small polarization $\chi_3 = 2{}^1\tilde{\Phi}/\tilde{\Phi} - (\tilde{\Lambda} + \Lambda)/{}^\mu\tilde{\Lambda} = 2\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)$. Re-expressing ${}^1\tilde{\Phi} = \tilde{\Phi}[(\tilde{\Lambda} + \Lambda)/2{}^\mu\tilde{\Lambda} + \underline{\zeta}\sin(\omega_0\varphi + \varphi_0)]$ and (88), one can generate a class of off-diagonal jet prolongations of ellipsoid Kerr – de Sitter configurations

$$\begin{aligned}
ds^2 = & e^{\psi(x^{k'})}(1 + \varepsilon\chi(x^{k'}))[(dx^{1'})^2 + (dx^{2'})^2] - \frac{\tilde{\Phi}^2\bar{A}}{4} \frac{1}{\mu\tilde{\Lambda}} [1 + 2\varepsilon\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)][dy^{3'} + (\partial_{k'} {}^\eta n(x^{i'}) - \partial_{k'}(\hat{y}^{3'} + \varphi\frac{\bar{B}}{\bar{A}}))dx^{k'}]^2 \\
& + \frac{(\partial_\varphi\tilde{\Phi})^2}{\mu\tilde{\Lambda}\tilde{\Phi}^2}(\bar{C} - \frac{\bar{B}^2}{\bar{A}})[1 + \varepsilon(\frac{\partial_\varphi\tilde{\Phi}}{\tilde{\Phi}}\frac{\tilde{\Lambda} + \Lambda}{\mu\tilde{\Lambda}} + 2\frac{\partial_\varphi\tilde{\Phi}}{\tilde{\Phi}}\underline{\zeta}\sin(\omega_0\varphi + \varphi_0) + 2\omega_0\underline{\zeta}\cos(\omega_0\varphi + \varphi_0)][d\varphi + (\partial_{i'}\tilde{A} + \varepsilon\partial_{i'}{}^1\tilde{A})dx^{i'}]^2 \\
& + \frac{{}^1\tilde{\Phi}^2}{4(\tilde{\Lambda} + \Lambda)}[d\zeta^5 + (\partial_\tau{}^1n)du^\tau]^2 + \frac{(\partial_6{}^1\tilde{\Phi})^2}{(\tilde{\Lambda} + \Lambda){}^1\tilde{\Phi}^2}[d\zeta^6 + (\partial_\tau{}^1\tilde{A})du^\tau]^2 \\
& + \frac{{}^2\tilde{\Phi}^2}{4(\tilde{\Lambda} + \Lambda)}[d\zeta^7 + (\partial_{\tau_1}{}^2n)du^{\tau_1}]^2 + \frac{(\partial_8{}^2\tilde{\Phi})^2}{(\tilde{\Lambda} + \Lambda){}^2\tilde{\Phi}^2}[d\zeta^8 + (\partial_{\tau_1}{}^2\tilde{A})du^{\tau_1}]^2.
\end{aligned}$$

These metrics possess the Killing symmetry ∂_7 and define ε -deformations of Kerr – de Sitter black holes into ellipsoid configurations with effective cosmological constants determined, respectively, by constants in Ricci soliton models, massive gravity, f -modifications and jet prolongation contributions.

A N-adapted Coefficients and Proofs

We provide a set of necessary N-adapted coefficient formulae that are important for proofs and applications. A series of results obtained in [40, 43, 45] are reformulated and generalized for nonholonomic r -jet variables with conventional $2 + 2 + \dots$ splitting.

A.1 Torsions and Curvature of d-connections on $J^r(V, V')$ with 2-d shells

For any d-connection structure ${}^s\mathbf{D}$ and r -jet 2d shell prolongations with coefficients (12), there are two important theorems:

Theorem A.1 *The N-adapted coefficients of d-torsion ${}^s\mathbf{T} = \{\mathbf{T}_{\beta_s\gamma_s}^{\alpha_s}\}$ from (15) are computed recurrently "shell by shell" following formulae*

$$\begin{aligned} T_{jk}^i &= L_{jk}^i - L_{kj}^i, T_{ja}^i = C_{jb}^i, T_{ji}^a = -{}^N J_{ji}^a, T_{aj}^c = L_{aj}^c - \partial_a(N_j^c), T_{bc}^a = C_{bc}^a - C_{cb}^a, \text{ spacetime components}; \\ T_{j_s k_s}^{i_s} &= L_{j_s k_s}^{i_s} - L_{k_s j_s}^{i_s}, T_{j_s a_s}^{i_s} = C_{j_s b_s}^{i_s}, T_{j_s i_s}^{a_s} = -{}^N J_{j_s i_s}^{a_s}, T_{a_s j_s}^{c_s} = L_{a_s j_s}^{c_s} - \partial_{a_s}(N_{j_s}^{c_s}), T_{b_s c_s}^{a_s} = C_{b_s c_s}^{a_s} - C_{c_s b_s}^{a_s}, \text{ r-jet}. \end{aligned} \quad (\text{A.1})$$

Proof. The coefficients (A.4) are computed for any $\hat{\mathbf{D}} = \{\mathbf{T}_{\beta_s\gamma_s}^{\alpha_s}\}$ and N-adapted frames (9) and (10) using standard differential form calculus with (15) (or, in operator form, using the formula (13)).

□

Theorem A.2 *The N-adapted coefficients of d-curvature ${}^s\mathbf{R} = \{\mathbf{R}_{\beta_s\gamma_s\delta_s}^{\alpha_s}\}$ from (16) are computed recurrently "shell by shell" following formulae*

$$\begin{aligned} R_{hjk}^i &= \partial_k L_{hj}^i - \partial_j L_{hk}^i + L_{hj}^m L_{mk}^i - L_{hk}^m L_{mj}^i - C_{ha}^i {}^N J_{kj}^a, \\ R_{bjk}^a &= \partial_k L_{bj}^a - \partial_j L_{bk}^a + L_{bj}^c L_{ck}^a - L_{bk}^c L_{cj}^a - C_{bc}^a {}^N J_{kj}^c, \\ R_{jka}^i &= \partial_a L_{jk}^i - D_k C_{ja}^i + C_{jb}^i \hat{T}_{ka}^b, R_{bka}^c = e_a L_{bk}^c - D_k C_{ba}^c + C_{bd}^c T_{ka}^d, \\ R_{jbc}^i &= \partial_c C_{jb}^i - \partial_b C_{jc}^i + C_{jb}^h C_{hc}^i - C_{jc}^h C_{hb}^i, \\ R_{bcd}^a &= \partial_d C_{bc}^a - \partial_c C_{bd}^a + C_{bc}^e C_{ed}^a - C_{bd}^e C_{ec}^a, \text{ spacetime components}; \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} R_{h_s j_s k_s}^{i_s} &= \partial_{k_s} L_{h_s j_s}^{i_s} - \partial_{j_s} L_{h_s k_s}^{i_s} + L_{h_s j_s}^{m_s} L_{m_s k_s}^{i_s} - L_{h_s k_s}^{m_s} L_{m_s j_s}^{i_s} - C_{h_s a_s}^{i_s} {}^N J_{k_s j_s}^{a_s}, \\ R_{b_s j_s k_s}^{a_s} &= \partial_{k_s} L_{bj}^a - \partial_j L_{bk}^a + L_{bj}^c L_{ck}^a - L_{bk}^c L_{cj}^a - C_{bc}^a {}^N J_{kj}^c, \\ R_{j_s k_s a_s}^{i_s} &= \partial_{a_s} L_{j_s k_s}^{i_s} - D_{k_s} C_{j_s a_s}^{i_s} + C_{j_s b_s}^{i_s} \hat{T}_{k_s a_s}^{b_s}, R_{b_s k_s a_s}^{c_s} = \partial_{a_s} L_{b_s k_s}^{c_s} - D_{k_s} C_{b_s a_s}^{c_s} + C_{b_s d_s}^{c_s} T_{k_s a_s}^d, \\ R_{j_s b_s c_s}^{i_s} &= \partial_{c_s} C_{j_s b_s}^{i_s} - \partial_{b_s} C_{j_s c_s}^{i_s} + C_{j_s b_s}^{h_s} C_{h_s c_s}^{i_s} - C_{j_s c_s}^{h_s} C_{h_s b_s}^{i_s}, \\ R_{b_s c_s d_s}^{a_s} &= \partial_{d_s} C_{b_s c_s}^{a_s} - \partial_{c_s} C_{b_s d_s}^{a_s} + C_{b_s c_s}^{e_s} C_{e_s d_s}^{a_s} - C_{b_s d_s}^{e_s} C_{e_s c_s}^{a_s}, \text{ spacetime components}. \end{aligned}$$

Proof. The coefficients (A.2) are computed for any $\hat{\mathbf{D}} = \{\mathbf{T}_{\beta_s\gamma_s}^{\alpha_s}\}$ and N-adapted frames (9) and (10) using a standard differential form calculus with (16) (or, in operator form, using the formula (14)). For $2 + 2 + \dots$ shell decompositions, such formulae are similar to the coefficients of curvature in higher dimensional spacetime but with re-parameterized (in our case) nonholonomic r -jet variables.

□

A.2 Sketch of proof of theorem 2.3

We can check by straightforward computations that the conditions of metric compatibility and zero h - and v - torsions are satisfied by ${}^s\hat{\mathbf{D}} = \{\hat{\Gamma}_{\alpha_s\beta_s}^{\gamma_s}\}$ with coefficients computed recurrently

$$\begin{aligned}
\hat{L}_{jk}^i &= \frac{1}{2}g^{ir}(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}), \\
\hat{L}_{bk}^a &= e_b(N_k^a) + \frac{1}{2}h^{ac}(\mathbf{e}_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d), \\
\hat{C}_{jc}^i &= \frac{1}{2}g^{ik}e_c g_{jk}, \quad \hat{C}_{bc}^a = \frac{1}{2}h^{ad}(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}), \\
\hat{L}_{\beta\gamma}^\alpha &= \frac{1}{2}g^{\alpha\tau}(\mathbf{e}_\gamma g_{\beta\tau} + \mathbf{e}_\beta g_{\gamma\tau} - \mathbf{e}_\tau g_{\beta\gamma}), \\
\hat{L}_{b_1\gamma}^{a_1} &= \partial_{b_1}(N_\gamma^{a_1}) + \frac{1}{2}h^{a_1 c_1}(\mathbf{e}_\gamma h_{b_1 c_1} - h_{d_1 c_1} \partial_{b_1} N_\gamma^{d_1} - h_{d_1 b_1} \partial_{c_1} N_\gamma^{d_1}), \\
\hat{C}_{\beta c_1}^\alpha &= \frac{1}{2}g^{\alpha\tau} \partial_{c_1} g_{\beta\tau}, \quad \hat{C}_{b_1 c_1}^{a_1} = \frac{1}{2}h^{a_1 d_1}(\partial_{c_1} h_{b_1 d_1} + \partial_{c_1} h_{c_1 d_1} - \partial_{d_1} h_{b_1 c_1}), \\
&\dots \\
\hat{L}_{\beta_{s-1}\gamma_{s-1}}^{\alpha_{s-1}} &= \frac{1}{2}g^{\alpha_{s-1}\tau_{s-1}}(\mathbf{e}_{\gamma_{s-1}} g_{\beta_{s-1}\tau_{s-1}} + \mathbf{e}_{\beta_{s-1}} g_{\gamma_{s-1}\tau_{s-1}} - \mathbf{e}_{\tau_{s-1}} g_{\beta_{s-1}\gamma_{s-1}}), \\
\hat{L}_{b_s\gamma_{s-1}}^{a_s} &= \partial_{b_s}(N_{\gamma_{s-1}}^{a_s}) + \frac{1}{2}h^{a_s c_s}(\mathbf{e}_{\gamma_{s-1}} h_{b_s c_s} - h_{d_s c_s} \partial_{b_s} N_{\gamma_{s-1}}^{d_s} - h_{d_s b_s} \partial_{c_s} N_{\gamma_{s-1}}^{d_s}), \\
\hat{C}_{\beta_{s-1}c_s}^{\alpha_{s-1}} &= \frac{1}{2}g^{\alpha_{s-1}\tau_{s-1}} \partial_{c_s} g_{\beta_{s-1}\tau_{s-1}}, \quad \hat{C}_{b_s c_s}^{a_s} = \frac{1}{2}h^{a_s d_s}(\partial_{c_s} h_{b_s d_s} + \partial_{c_s} h_{c_s d_s} - \partial_{d_s} h_{b_s c_s}).
\end{aligned} \tag{A.3}$$

The torsion d-tensor (15) of ${}^s\hat{\mathbf{D}}$ is completely defined by ${}^s\mathbf{g}$ (18) for any chosen ${}^s\mathbf{N} = \{N_{i_s}^{a_s}\}$ if the above coefficients (A.3) are introduced "shell by shell" into formulae

$$\begin{aligned}
\hat{T}_{jk}^i &= \hat{L}_{jk}^i - \hat{L}_{kj}^i, \quad \hat{T}_{ja}^i = \hat{C}_{jb}^i, \quad \hat{T}_{ji}^a = -{}^N J_{ji}^a, \quad \hat{T}_{aj}^c = \hat{L}_{aj}^c - e_a(N_j^c), \quad \hat{T}_{bc}^a = \hat{C}_{bc}^a - \hat{C}_{cb}^a, \\
&\dots \\
\hat{T}_{\beta_s\gamma_s}^{\alpha_s} &= \hat{L}_{\beta_s\gamma_s}^{\alpha_s} - \hat{L}_{\gamma_s\beta_s}^{\alpha_s}, \quad \hat{T}_{\beta_s b_s}^{\alpha_s} = \hat{C}_{\beta_s b_s}^{\alpha_s}, \quad \hat{T}_{\beta_s \gamma_s}^{a_s} = {}^N J_{\gamma_s \beta_s}^{a_s}.
\end{aligned} \tag{A.4}$$

We can impose as additional nonholonomic conditions certain equations when all coefficients (A.4) are zero. In this case we extract from (A.3) various LC-configurations. The coefficients of the LC-connection ${}_1\Gamma_{\alpha_s\beta_s}^{\gamma_s}$ can be computed in standard form in coordinate bases and/or with respect to N-adapted frames. Taking differences between $\hat{\Gamma}_{\alpha_s\beta_s}^{\gamma_s}$ and ${}_1\Gamma_{\alpha_s\beta_s}^{\gamma_s}$, we find the N-adapted coefficients of the distortion d-tensor $\hat{\mathbf{Z}}_{\beta_s\gamma_s}^{\alpha_s}$ (similar formulae are given for Corollary 2.1 and (22) in Ref. [43], in extra dimension coordinates but without jet configurations).

A.3 Proof of theorem 3.1

Such a proof is possible by explicitly the computing of the N-adapted coefficients of the canonical Ricci d-tensor on $\mathbf{J}^r(\mathbf{V}, \mathbf{V}')$ with 2-d shells. Let us consider an ansatz (40) with $\partial_4 h_a \neq 0, \partial_6 h_{a_1} \neq 0, \dots, \partial_{2s} h_{a_s} \neq 0$, when the partial derivatives are denoted in the forms $\partial_1 h = \partial h / \partial x^1$, $\partial_4 h = \partial h / \partial y^4$, $\partial_{44}^2 h = \partial^2 h / \partial y^4 \partial y^4$ and $\partial_{66}^2 = \partial^2 h / \partial \zeta^6 \partial \zeta^6$, where the indices taking values 5, 6, ... are for $2 + 2 + \dots$ jet parameterized variables. We can construct more special classes of solutions when the conditions alluded to are not satisfied which warrants the analysis of more special classes of solutions. For simplicity, we suppose that via frame transformations it is always possible to introduce the necessary type of parameterizations for d-metrics whenever the necessary types of partial derivatives of some coefficients are not zero.

Lemma A.1 *With respect to N-adapted frames (9) and (10), the nonzero coefficients of the Ricci d-tensor $\hat{\mathbf{R}}_{\alpha_s\beta_s}$ (21) for ansatz (40) with Killing symmetry on ∂_3 possess symmetries determined by the following formulae: for*

spacetime components with partial derivative operator ∂ ,

$$\hat{R}_1^1 = \hat{R}_2^2 = -\frac{1}{2g_1g_2}[\partial_{11}^2g_2 - \frac{(\partial_1g_1)(\partial_1g_2)}{2g_1} - \frac{(\partial_1g_2)^2}{2g_2} + \partial_{22}^2g_1 - \frac{(\partial_2g_1)(\partial_2g_2)}{2g_2} - \frac{(\partial_2g_1)^2}{2g_1}], \quad (\text{A.5})$$

$$\hat{R}_3^3 = \hat{R}_4^4 = -\frac{1}{2h_3h_4}[\partial_{44}^2h_3 - \frac{(\partial_4h_3)^2}{2h_3} - \frac{(\partial_4h_3)(\partial_4h_4)}{2h_4}], \quad (\text{A.6})$$

$$\hat{R}_{3k} = \frac{h_3}{2h_4}\partial_{44}^2n_k + \left(\frac{h_3}{h_4}\partial_4h_4 - \frac{3}{2}\partial_4h_3\right)\frac{\partial_4n_k}{2h_4}, \quad (\text{A.7})$$

$$\hat{R}_{4k} = \frac{w_k}{2h_3}[\partial_{44}^2h_3 - \frac{(\partial_4h_3)^2}{2h_3} - \frac{(\partial_4h_3)(\partial_4h_4)}{2h_4}] + \frac{\partial_4h_3}{4h_3}\left(\frac{\partial_kh_3}{h_3} + \frac{\partial_kh_4}{h_4}\right) - \frac{\partial_k(\partial_4h_3)}{2h_3}, \quad (\text{A.8})$$

and for r -jet components with partial derivative operator ∂ on jet variables, on shell $s = 1, \tau = 1, 2, 3, 4$;

$$\hat{R}_5^5 = \hat{R}_6^6 = -\frac{1}{2h_5h_6}[\partial_{66}^2h_5 - \frac{(\partial_6h_5)^2}{2h_5} - \frac{(\partial_6h_5)(\partial_6h_6)}{2h_6}], \quad (\text{A.9})$$

$$\hat{R}_{5\tau} = \frac{h_5}{2h_6}\partial_{66}^2n_\tau + \left(\frac{h_5}{h_6}\partial_6h_6 - \frac{3}{2}\partial_6h_5\right)\frac{\partial_6n_\tau}{2h_6}, \quad (\text{A.10})$$

$$\hat{R}_{6\tau} = \frac{w_\tau}{2h_5}[\partial_{66}^2h_5 - \frac{(\partial_6h_5)^2}{2h_5} - \frac{(\partial_6h_5)(\partial_6h_6)}{2h_6}] + \frac{\partial_6h_5}{4h_5}\left(\frac{\partial_\tau h_5}{h_5} + \frac{\partial_\tau h_6}{h_6}\right) - \frac{\partial_\tau(\partial_6h_5)}{2h_5}, \quad (\text{A.11})$$

and, for extra shells with number s ,

$$\begin{aligned} \hat{R}_{2s-1}^{2s-1} &= \hat{R}_{2s}^{2s} = -\frac{1}{2h_{3+2s}h_{4+2s}}[\partial_{4+2s}^2h_{3+2s} - \frac{(\partial_{4+2s}h_{3+2s})^2}{2h_{3+2s}} - \frac{(\partial_{4+2s}h_{3+2s})(\partial_{4+2s}h_{4+2s})}{2h_{4+2s}}], \\ \hat{R}_{3+2s}^{2s-1} &= \frac{h_{2s-1}}{2h_{2s}}\partial_{2s}^2n_{\tau_1} + \left(\frac{h_{2s-1}}{h_{2s}}\partial_{4+2s}h_{4+2s} - \frac{3}{2}\partial_{4+2s}h_{3+2s}\right)\frac{\partial_{4+2s}n_{\tau_1}}{2h_{3+2s}}, \\ \hat{R}_{4+2s}^{2s-1} &= \frac{w_{\tau_1}}{2h_7}[\partial_{4+2s}^2h_{3+2s} - \frac{(\partial_{4+2s}h_{3+2s})^2}{2h_{3+2s}} - \frac{(\partial_{4+2s}h_{3+2s})(\partial_{4+2s}h_{4+2s})}{2h_{4+2s}}] + \\ &\quad \frac{\partial_{4+2s}h_{3+2s}}{4h_{3+2s}}\left(\frac{\partial_{\tau_1}h_{3+2s}}{h_{3+2s}} + \frac{\partial_{\tau_1}h_{4+2s}}{h_{4+2s}}\right) - \frac{\partial_{\tau_1}(\partial_{4+2s}h_{3+2s})}{2h_{3+2s}}, \end{aligned} \quad (\text{A.12})$$

when $\tau_1 = 1, 2, 3, 4, 5, 6$;

...

Proof. We introduce the coefficients of the canonical d-connection $\hat{\Gamma}_{\alpha_s\beta_s}^{\gamma_s}$ (A.3) for the d-metric ansatz (40) and compute the N-adapted d-curvature coefficients (A.2) and (A.1). Then, contracting the indices (following formulae (21), (22) and (23)) we find the nontrivial values of the N-adapted coefficients for the Ricci d-tensor, scalar curvature and Einstein d-tensor of ${}^s\hat{\mathbf{D}}$. Explicit proofs of the formulae (A.5)–(A.8) for 4-d and extra dimensional indices are provided in a series of our works, for instance, in [45, 43, 40]. We do not repeat the required calculus in this paper.

Introducing r -jet variables, we observe that on the first shell, with $s = 1$, the formulae (A.6)–(A.8) are generalized in a similar form but for the partial derivatives ∂ on jet variables, with respective indices 5 and 6 for a nonholonomic $2+2+2+\dots$ splitting. To avoid ambiguities, we put left labels $s = 1$ on the necessary geometric objects and coefficients. On this shell, the first four coordinates $\alpha = 1, 2, 3, 4$ are treated as "base type" but take $a_1, b_1, \dots = 5, 6$ as conventional "fiber/jet" ones. In symbolic form, the equations (A.9)–(A.11) are constructed via formally increasing by 2 respective values of 4-d spacetime indices and introducing dependencies on all "base/spacetime" coordinates.

For shells $s = 2, 3, \dots$, "fiber/jet" indices are labeled with values of type $3 + 2s$ and $4 + 2s$ and the previous (base type) indices take values $1, 2, \dots, 2 + 2s$. The equations (A.12) present a "recurrent" generalizations for a finite number of shells, s , of the 1st jet shell when $s = 1$.

□

Let us analyze some important nonholonomic symmetries of the canonical Ricci and Einstein d-tensors:

For $s = 1$ and using the above formulae, we can compute the Ricci scalar (22) for ${}^1\hat{\mathbf{D}}$, ${}^1\hat{R} = 2(\hat{R}_1^1 + \hat{R}_3^3 + \hat{R}_5^5)$. There are certain N-adapted symmetries of the Einstein d-tensor (23) for the ansatz (40), $\hat{E}_1^1 = \hat{E}_2^2 = -(\hat{R}_3^3 + \hat{R}_5^5)$, $\hat{E}_3^3 = \hat{E}_4^4 = -(\hat{R}_1^1 + \hat{R}_5^5)$, $\hat{E}_5^5 = \hat{E}_6^6 = -(\hat{R}_1^1 + \hat{R}_3^3)$.

In a similar form, we find symmetries for $s = 2$:

$$\begin{aligned}\hat{E}_1^1 &= \hat{E}_2^2 = -(\hat{R}_3^3 + \hat{R}_5^5 + \hat{R}_7^7), \hat{E}_3^3 = \hat{E}_4^4 = -(\hat{R}_1^1 + \hat{R}_5^5 + \hat{R}_7^7), \\ \hat{E}_5^5 &= \hat{E}_6^6 = -(\hat{R}_1^1 + \hat{R}_3^3 + \hat{R}_7^7), \hat{E}_7^7 = \hat{E}_8^8 = -(\hat{R}_1^1 + \hat{R}_3^3 + \hat{R}_5^5).\end{aligned}$$

We conclude that the nonholonomically jet modified Einstein equations (A.5)–(A.12) for $s = 2$ jet shells with nontrivial Λ -sources can be written in N-adapted form as

$$\hat{R}_1^1 = \hat{R}_2^2 = -\Lambda(x^k), \hat{R}_3^3 = \hat{R}_4^4 = -{}^v\Lambda(x^k, y^4), \hat{R}_5^5 = \hat{R}_6^6 = -{}^v\Lambda(u^\beta, \zeta^6), \hat{R}_7^7 = \hat{R}_8^8 = -{}^v\Lambda(u^{\beta_1}, \zeta^8), \dots, \quad (\text{A.13})$$

which can be extended for any arbitrary finite number of jets' shells.

A.4 Nonholonomic spacetime and r -jet vacuum solutions

A.4.1 4-d nonholonomic vacuum configurations

To consider vacuum solutions for $\hat{\mathbf{D}}$ with ${}^v\Lambda = 0$ in (47) we study configurations with N-adapted coefficients when $\partial_4 h_3 = 0$ and/or $\partial_4 \phi = 0$. The limits to the off-diagonal solutions with $\Lambda = {}^v\Lambda = 0$ are not smooth because multiples $({}^v\Lambda)^{-1}$ are considered in various coefficients and re-defined generating functions for solutions (61).

Let us analyze the conditions when the nontrivial coefficients of the Ricci d-tensor (A.5)–(A.8) are zero for ansatz (40). The first equation is a typical example of 2-d wave or Laplace equation. We can express such solutions in a similar form $g_i = \epsilon_i e^{\psi(x^k, \Lambda=0)} (dx^i)^2$.

There are three classes of off-diagonal metrics giving zero coefficients (A.6)–(A.8).

1. We impose the condition $\partial_4 h_3 = 0, h_3 \neq 0$, giving only one nontrivial equation, see (A.7), $\partial_{44}^2 n_k + \partial_4 n_k \partial_4 \ln |h_4| = 0$, where $h_4(x^i, y^4) \neq 0$ and $w_k(x^i, y^4)$ are arbitrary functions. If $\partial_4 h_4 = 0$, we must take $\partial_{44}^2 n_k = 0$. For $\partial_4 h_4 \neq 0$, we get

$$n_k = {}_1n_k + {}_2n_k \int dy^4 / h_4 \quad (\text{A.14})$$

with integration functions ${}_1n_k(x^i)$ and ${}_2n_k(x^i)$. The corresponding class of nonholonomic vacuum solutions is defined by quadratic line element

$$\begin{aligned}ds_{v1}^2 &= \epsilon_i e^{\psi(x^k, \Lambda=0)} (dx^i)^2 + {}^0h_3(x^k) [dy^3 + ({}_1n_k(x^i) + {}_2n_k(x^i) \int dy^4 / h_4) dx^i]^2 \\ &\quad + h_4(x^i, y^4) [dy^4 + w_i(x^k, y^4) dx^i].\end{aligned}$$

2. Let us assume $\partial_4 h_3 \neq 0$ and $\partial_4 h_4 \neq 0$. We can solve (A.6) and/or (47) for ${}^v\Lambda = 0$ if $\partial_4 \phi = 0$ for coefficients (42) and (41). For $\phi = \phi_0 = \text{const}$, we can consider arbitrary functions $w_i(x^k, y^4)$ as generating functions because $\beta = \alpha_i = 0$ for such configurations. The condition (41) is satisfied by any

$$h_4 = {}^0h_4(x^k) (\partial_4 \sqrt{|h_3|})^2, \quad (\text{A.15})$$

where ${}^0h_3(x^k)$ is an integration function and $h_3(x^k, y^4)$ is any generating function. The coefficients n_k are found from (A.7), see (A.14). The corresponding class of nonholonomic vacuum metrics is defined by the quadratic line element

$$\begin{aligned}ds_{v2}^2 &= \epsilon_i e^{\psi(x^k, \Lambda=0)} (dx^i)^2 + h_3(x^i, y^4) [dy^3 + ({}_1n_k(x^i) + {}_2n_k(x^i) \int dy^4 / h_4) dx^i]^2 + \\ &\quad {}^0h_4(x^k) (\partial_4 \sqrt{|h_3|})^2 [dy^4 + w_i(x^k, y^4) dx^i]^2.\end{aligned} \quad (\text{A.16})$$

3. Another type of configurations are generated by $\partial_4 h_3 \neq 0$ but $\partial_4 h_4 = 0$. The equation (A.6) is $\partial_4^2 h_3 - \frac{(\partial_4 h_3)^2}{2h_3} = 0$, with general solution is $h_3(x^k, y^4) = [c_1(x^k) + c_2(x^k)y^4]^2$, where $c_1(x^k), c_2(x^k)$ are generating functions and $h_4 = {}^0h_4(x^k)$. For $\phi = \phi_0 = \text{const}$, we can choose any values $w_i(x^k, y^4)$ because $\beta = \alpha_i = 0$. The coefficients n_i are determined by equation (A.7) and/or, equivalently, (48) with $\gamma = \frac{3}{2}\partial_4|h_3|$. We find

$$n_i = {}_1n_i(x^k) + {}_2n_i(x^k) \int dy^4 |h_3|^{-3/2} = {}_1n_i(x^k) + {}_2\tilde{n}_i(x^k)[c_1(x^k) + c_2(x^k)y^4]^{-2},$$

with integration functions ${}_1n_i(x^k)$ and ${}_2n_i(x^k)$, or re-defined ${}_2\tilde{n}_i = -{}_2n_i/2c_2$. The quadratic line element for this class of vacuum nonholonomic solutions is given by

$$ds_{v3}^2 = \epsilon_i e^{\psi(x^k, \Lambda=0)} (dx^i)^2 + [c_1(x^k) + c_2(x^k)y^4]^2 [dy^3 + ({}_1n_i(x^k) + {}_2\tilde{n}_i(x^k)[c_1(x^k) + c_2(x^k)y^4]^{-2}) dx^i]^2 + {}^0h_4(x^k) [dy^4 + w_i(x^k, y^4) dx^i]^2. \quad (\text{A.17})$$

Finally, we note that such solutions are with nontrivial induced torsion (A.4) and that additional assumptions are necessary to extract vacuum LC-configurations.

A.4.2 Nonholonomic r -jet prolongations of vacuum solutions:

The quadratic line elements (61), (62), (63),... for off-diagonal jet prolongations of generic off-diagonal solutions have been constructed for nontrivial sources ${}_v\Lambda(x^k, y^4)$, ${}_1\Lambda(u^\tau, \zeta^6)$, ${}_2\Lambda(u^\tau, \zeta^8)$,... In a similar manner, we can generate jet prolongations of vacuum configurations with effective zero cosmological constants extending with r -jet variables the 4-d vacuum metrics of type ds_{v1}^2 , ds_{v2}^2 (A.16), ds_{v3}^2 (A.17) etc. It is possible to generate solutions when the sources are zero on some shells and nonzero on other shells.

Let us consider an example of quadratic line element for jet prolongation of effective 6-d gravity derived as a $s = 1$ generalization of (A.16). For such solutions, $\partial_4 h_a \neq 0, \partial_6 h_{a_1} \neq 0, \dots$ and $\phi = \phi_0 = \text{const}$, ${}^1\phi = {}^1\phi_0 = \text{const}, \dots$

$$ds_{v2s3}^2 = \epsilon_i e^{\psi(x^k, \Lambda=0)} (dx^i)^2 + h_3(x^i, y^4) [dy^3 + ({}_1n_k(x^i) + {}_2n_k(x^i) \int dy^4 / h_4) dx^i]^2 + {}^0h_4(x^k) (\partial_4 \sqrt{|h_3|})^2 [dy^4 + w_i(x^k, y^4) dx^i]^2 + h_5(u^\tau, \zeta^6) [d\zeta^5 + ({}_1n_\lambda(u^\tau) + {}_2n_\lambda(u^\tau) \int d\zeta^6 / h_6) du^\lambda]^2 + {}^0h_6(u^\tau) (\partial_6 \sqrt{|h_5|})^2 [d\zeta^6 + {}^1w_\lambda(u^\tau, \zeta^6) du^\lambda]^2, \quad (\text{A.18})$$

where ${}^0h_3(x^k)$, ${}^0h_5(u^\tau)$, ${}_1n_k(x^i)$, ${}_2n_k(x^i)$, ${}_1n_\lambda(u^\tau)$, ${}_2n_\lambda(u^\tau)$ are integration functions. The values $h_4(x^k, y^4)$ and $h_6(u^\tau, \zeta^6)$ are any generating functions depending on spacetime and jet prolongation variables. We can consider arbitrary functions $w_i(x^k, y^4)$ and ${}^1w_\lambda(u^\tau, \zeta^6)$ because, respectively, $\beta = \alpha_i = 0$ and ${}^1\beta = {}^1\alpha_\tau = 0$ for such configurations, see formulas (42), (41) and (44), (43).

A.5 The LC-conditions

We can consider nonholonomic frame deformations of the N-coefficients and ansatz (40) when all coefficients of a nonholonomically induced torsion (A.4) are zero and ${}_1\Gamma_{\alpha_s \beta_s}^{\gamma_s} = \widehat{\Gamma}_{\alpha_s \beta_s}^{\gamma_s}$. For simplicity, we analyze such conditions for 4-d spacetime (generalizations to extra jet shell can be performed recurrently as we explained in section 3).

The trivial coefficients of d-torsion (A.4) are $\widehat{T}_{jk}^i = \widehat{L}_{jk}^i - \widehat{L}_{kj}^i = 0$, $\widehat{T}_{ja}^i = \widehat{C}_{ja}^i = 0$, $\widehat{T}_{bc}^a = \widehat{C}_{bc}^a - \widehat{C}_{cb}^a = 0$ for any ansatz (40). Let us compute the nontrivial coefficients $\widehat{T}_{aj}^c = \widehat{L}_{aj}^c - e_a(N_j^c)$ and $\widehat{T}_{ji}^a = -{}^N J_{ji}^a$. For a 2+2 spacetime splitting, the values

$$\widehat{L}_{bi}^a = \partial_b N_i^a + \frac{1}{2} h^{ac} (\partial_i h_{bc} - N_i^e \partial_e h_{bc} - h_{dc} \partial_b N_i^d - h_{db} \partial_c N_i^d), \quad \widehat{T}_{aj}^c = \frac{1}{2} h^{ac} (\partial_i h_{bc} - N_i^e \partial_e h_{bc} - h_{dc} \partial_b N_i^d - h_{db} \partial_c N_i^d).$$

are computed for $N_i^3 = n_i(x^k, y^4)$, $N_i^4 = w_i(x^k, y^4)$; $h_{bc} = \text{diag}[h_3(x^k, y^4), h_4(x^k, y^4)]$; $h^{ac} = \text{diag}[(h_3)^{-1}, (h_4)^{-1}]$. We write

$$\begin{aligned}\hat{T}_{bi}^3 &= \frac{1}{2}h^{3c}(\partial_i h_{bc} - N_i^e \partial_e h_{bc} - h_{dc} \partial_b N_i^d - h_{db} \partial_c N_i^d) = \frac{1}{2h_3}(\partial_i h_{b3} - w_i \partial_4 h_{b3} - h_3 \partial_b n_i), \\ \text{i.e. } \hat{T}_{3i}^3 &= \frac{1}{2h_3}(\partial_i h_3 - w_i \partial_4 h_3), \quad \hat{T}_{4i}^3 = \frac{1}{2} \partial_4 n_i.\end{aligned}$$

In a similar form, we compute

$$\begin{aligned}\hat{T}_{bi}^4 &= \frac{1}{2}h^{4c}(\partial_i h_{bc} - N_i^e \partial_e h_{bc} - h_{dc} \partial_b N_i^d - h_{db} \partial_c N_i^d) = \frac{1}{2h_4}(\partial_i h_{b4} - w_i \partial_4 h_{b4} - h_4 \partial_b w_i - h_{3b} \partial_4 n_i - h_{4b} \partial_4 w_i) \\ \text{i.e. } \hat{T}_{3i}^4 &= -\frac{h_3}{2h_4} \partial_4 n_i, \quad \hat{T}_{4i}^4 = \frac{1}{2h_4}(\partial_i h_4 - w_i \partial_4 h_4) - \partial_4 w_i.\end{aligned}$$

The coefficients of the N-connection curvature ${}^N J_{ij}^a = \mathbf{e}_j(N_i^a) - \mathbf{e}_i(N_j^a)$ are expressed as

$${}^N J_{ij}^a = \partial_j(N_i^a) - \partial_i(N_j^a) - N_j^b \partial_b N_i^a + N_i^b \partial_b N_j^a = \partial_j(N_i^a) - \partial_i(N_j^a) - w_j \partial_4 N_i^a + w_i \partial_4 N_j^a$$

with nontrivial values:

$${}^N J_{12}^3 = -{}^N J_{21}^3 = \partial_2 n_1 - \partial_1 n_2 - w_2 \partial_4 n_1 + w_1 \partial_4 n_2, \quad {}^N J_{12}^4 = -{}^N J_{21}^4 = \partial_2 w_1 - \partial_1 w_2 - w_2 \partial_4 w_1 + w_1 \partial_4 w_2. \quad (\text{A.19})$$

Summarizing the above formulae for $\partial_4 n_i = 0$ and $\partial_2 n_1 - \partial_1 n_2 = 0$, we get the condition for zero torsion for the ansatz (40) with $n_k = \partial_k n(x^i)$,

$$\frac{1}{2h_3}(\partial_i h_3 - w_i \partial_4 h_3) = 0, \quad \frac{1}{2h_4}(\partial_i h_4 - w_i \partial_4 h_4) = \partial_4 w_i, \quad (\text{A.20})$$

$$\partial_2 w_1 - \partial_1 w_2 - w_2 \partial_4 w_1 + w_1 \partial_4 w_2 = 0. \quad (\text{A.21})$$

From this, we can define a LC-configuration. The final step is to impose the condition that the coefficients n_k do not depend on y^4 . This can be fixed for ${}_1 n_k(x^i) = \partial_k n(x^i)$ and ${}_2 n_k = 0$, i.e. $n_k = \partial_k n(x^i)$.

Finally, we note that the LC-conditions can be formulated recurrently, in similar forms, for higher order shells of jet coordinates using the partial derivative operator $\tilde{\partial}$ both for zero and non-zero sources.

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